

AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

GRADIMIR V. MILOVANOVIĆ

ABSTRACT. Let W_n be the set of all algebraic polynomials of exact degree n whose coefficients are all nonnegative. For the norm in $L^2[0, \infty)$ with generalized Laguerre weight function $w(x) = x^\alpha e^{-x}$ ($\alpha > -1$), the extremal problem $C_n(\alpha) = \sup_{P \in W_n} (\|P'\|/\|P\|)^2$ is solved, which completes a result of A. K. Varma [7].

1. In this paper we give the complete solution of a problem which has been investigated recently by A. K. Varma (see [7, 8]). This problem is related to some previous integral inequalities of Varma [9, 10] and also to the classical inequalities of A. Markov [4], P. Erdős [1], G. G. Lorentz [2, 3], G. Szegő [5], and P. Turan [6].

Let W_n be the set of all algebraic polynomials of exact degree n , all coefficients of which are nonnegative, i.e.,

$$W_n = \left\{ P_n \mid P_n(x) = \sum_{k=0}^n a_k x^k, a_k \geq 0 \ (k = 0, 1, \dots, n) \right\}.$$

We denote by W_n^0 the subset of W_n for which $a_0 = 0$ (i.e., $P_n(0) = 0$).

Let $w(x) = x^\alpha e^{-x}$ ($\alpha > -1$) be a weight function on $[0, \infty)$, and let $\|f\|^2 = (f, f)$, where

$$(f, g) = \int_0^\infty w(x) f(x) g(x) dx \quad (f, g \in L^2[0, \infty)).$$

In [7] Varma has investigated the problem of determining the best constant in the inequality

$$(1.1) \quad \|P_n'\|^2 \leq C_n(\alpha) \|P_n\|^2,$$

where $P_n \in W_n$. In fact, he has proved

THEOREM A. *Let $P_n(x)$ be an algebraic polynomial of exact degree n with nonnegative coefficients. Then for $\alpha \geq (\sqrt{5} - 1)/2$,*

$$\int_0^\infty (P_n'(x))^2 x^\alpha e^{-x} dx \leq \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} \int_0^\infty P_n^2(x) x^\alpha e^{-x} dx,$$

equality holding for $P_n(x) = x^n$. For $0 \leq \alpha \leq 1/2$ we have

$$(1.2) \quad \int_0^\infty (P_n'(x))^2 x^\alpha e^{-x} dx \leq \frac{1}{(2 + \alpha)(1 + \alpha)} \int_0^\infty P_n^2(x) x^\alpha e^{-x} dx.$$

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Moreover, (1.2) is also best possible in the sense that for $P_n(x) = x^n + \lambda x$ the expression on the left can be made arbitrarily close to the expression on the right by choosing λ positive and sufficiently large.

The case $\alpha = 1$ was considered in [8]. The cases $\alpha \in (-1, 0)$ and $\alpha \in (1/2, (\sqrt{5} - 1)/2)$ are still open.

2. The object of this paper is to determine

$$(2.1) \quad C_n(\alpha) = \sup_{P_n \in W_n} \frac{\|P'_n\|^2}{\|P_n\|^2}$$

for all $\alpha \in (-1, \infty)$ and, thus, to give a complete solution of the extremal problem (1.1). Note that the supremum in (2.1) is attained for some $P_n \in W_n^0$. Indeed,

$$\sup_{P_n \in W_n} \frac{\|P'_n\|}{\|P_n\|} = \sup_{\substack{P_n \in W_n^0 \\ a_0 \geq 0}} \frac{\|P'_n\|}{\|P_n + a_0\|} = \sup_{P_n \in W_n^0} \frac{\|P'_n\|}{\|P_n\|}.$$

We begin by proving three lemmas:

LEMMA 1. If $P_n \in W_n$ then for every $x \geq 0$ the inequality

$$(2.2) \quad x(P'_n(x))^2 - P_n(x)P''_n(x) \leq P'_n(x)P_n(x)$$

holds.

PROOF. Let $P_n \in W_n$, i.e., $P_n(x) = \sum_{k=0}^n a_k x^k$ with $a_k \geq 0$ ($k = 0, 1, \dots, n$). Using the Cauchy-Schwarz inequality

$$\left| \sum_{k=0}^n x_k y_k \right|^2 \leq \left(\sum_{k=0}^n |x_k|^2 \right) \left(\sum_{k=0}^n |y_k|^2 \right)$$

for $x_k = a_k^{1/2} x^{k/2}$ and $y_k = k a_k^{1/2} x^{k/2}$ ($x \geq 0$), we obtain

$$\left(\sum_{k=0}^n k a_k x^k \right) \leq \left(\sum_{k=0}^n a_k x^k \right) \left(\sum_{k=0}^n k^2 a_k x^k \right),$$

which is equivalent to (2.2). \square

LEMMA 2. If $P_n \in W_n^0$, then for the integrals

$$J_n(\alpha) = \int_0^\infty x^\alpha e^{-x} P'_n(x)^2 dx,$$

$$I_{n,i}(\alpha) = \int_0^\infty x^\alpha e^{-x} P_n(x) P_n^{(i)}(x) dx \quad (i = 0, 1, 2)$$

the following recurrence relations hold:

$$I_{n,2}(\alpha) = I_{n,1}(\alpha) - \alpha I_{n,1}(\alpha - 1) - J_n(\alpha) \quad (\alpha > -1),$$

$$2I_{n,1}(\alpha) = I_{n,0}(\alpha) - \alpha I_{n,0}(\alpha - 1) \quad (\alpha > -2).$$

The proof of this lemma is a simple application of integration by parts and will be omitted. We note that the integrals $I_{n,1}(\alpha)$ and $I_{n,0}(\alpha - 1)$ exist for $\alpha > -2$ because $P_n(0) = 0$.

From Lemmas 1 and 2 there immediately follows

LEMMA 3. If $P_n \in W_n^0$, then for $\alpha > -1$,

$$J_n(\alpha) \leq \frac{1}{4} \left\{ I_{n,0}(\alpha) + (1 - 2\alpha)I_{n,0}(\alpha - 1) + (\alpha - 1)^2 I_{n,0}(\alpha - 2) \right\}.$$

THEOREM. The best constant $C_n(\alpha)$ defined in (2.1) is

$$(2.3) \quad C_n(\alpha) = \begin{cases} 1/(2 + \alpha)(1 + \alpha) & (-1 < \alpha \leq \alpha_n), \\ n^2/(2n + \alpha)(2n + \alpha - 1) & (\alpha_n \leq \alpha < +\infty), \end{cases}$$

where

$$(2.4) \quad \alpha_n = \frac{1}{2}(n + 1)^{-1} \left((17n^2 + 2n + 1)^{1/2} - 3n + 1 \right).$$

PROOF. Let $P_n \in W_n^0$, i.e., $P_n(x) = \sum_{k=1}^n a_k x^k$ ($a_k \geq 0$). Then

$$P_n(x)^2 = \sum_{k=2}^{2n} b_k x^k \quad (b_k \geq 0)$$

and

$$\|P_n\|^2 = I_{n,0}(\alpha) = \sum_{k=2}^{2n} b_k \Gamma(k + \alpha + 1),$$

where Γ is the gamma function. Using Lemma 3 we obtain

$$4J_n(\alpha) \leq \sum_{k=2}^{2n} b_k \left\{ \Gamma(k + \alpha + 1) + (1 - 2\alpha)\Gamma(k + \alpha) + (\alpha - 1)^2 \Gamma(k + \alpha - 1) \right\},$$

i.e.,

$$(2.5) \quad J_n(\alpha) \leq \sum_{k=2}^{2n} H_k(\alpha) b_k \Gamma(k + \alpha + 1),$$

where

$$H_k(\alpha) = \frac{1}{4} \cdot k^2 / (k + \alpha)(k + \alpha - 1).$$

From (2.5) it follows that

$$\|P'_n\|^2 \leq \left(\max_{2 \leq k \leq 2n} H_k(\alpha) \right) \|P_n\|^2,$$

so

$$C_n(\alpha) \leq \max_{2 \leq k \leq 2n} H_k(\alpha).$$

Determining the maximum of $f(x) = x^2/(x + \alpha)(x + \alpha - 1)$ on the interval $[2, 2n]$, we find that

$$\max_{2 \leq k \leq 2n} H_k(\alpha) = \begin{cases} H_2(\alpha) & \text{if } -1 < \alpha \leq \alpha_n, \\ H_{2n}(\alpha) & \text{if } \alpha_n \leq \alpha < +\infty, \end{cases}$$

where α_n is given by (2.4).

In order to show that $C_n(\alpha)$ defined in (2.3) is best possible, i.e. that $C_n(\alpha) = \max_{2 \leq k \leq 2n} H_k(\alpha)$, we consider $\tilde{P}_n(x) = x^n + \lambda x$ ($\lambda \geq 0$) and set

$$Q_n(\lambda) = \|\tilde{P}'_n\|^2 / \|\tilde{P}_n\|^2.$$

By a simple computation we find that

$$Q_n(\lambda) = \frac{n^2 \Gamma(2n + \alpha - 1) + 2\lambda n \Gamma(n + \alpha) + \lambda^2 \Gamma(\alpha + 1)}{\Gamma(2n + \alpha + 1) + 2\lambda \Gamma(n + \alpha + 2) + \lambda^2 \Gamma(\alpha + 3)}.$$

Since

$$Q_n(0) = n^2 / (2n + \alpha)(2n + \alpha - 1) = H_{2n}(\alpha)$$

and

$$\lim_{\lambda \rightarrow \infty} Q_n(\lambda) = 1/(\alpha + 1)(\alpha + 2) = H_2(\alpha),$$

we conclude that $\tilde{P}_n(x) = x^n$ is an extremal polynomial for $\alpha \geq \alpha_n$. Furthermore, if $-1 < \alpha \leq \alpha_n$, there exists a sequence of polynomials, for example, $p_{n,k}(x) = x^n + kx$, $k = 1, 2, \dots$, for which

$$\lim_{k \rightarrow \infty} \frac{\|p'_{n,k}\|^2}{\|p_{n,k}\|^2} = C_n(\alpha). \quad \square$$

REMARK. From (2.4) we have $\alpha_1 = (\sqrt{5} - 1)/2$, $\alpha_2 = (\sqrt{73} - 5)/6$, $\alpha_3 = (\sqrt{10} - 2)/2$, etc. The sequence (α_k) is decreasing, i.e., $\alpha_1 > \alpha_2 > \alpha_3 > \dots > \alpha_\infty$, where $\alpha_\infty = \lim_{n \rightarrow \infty} \alpha_n = (\sqrt{17} - 3)/2 \cong 0.56155$.

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FACULTY OF ELECTRONIC ENGINEERING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIŠ, BE-GRADSKA 14, P. O. BOX 73 18000 NIŠ, YUGOSLAVIA