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A MONOTONE PRINCIPLE OF FIXED POINTS

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ABSTRACT. In this paper we formulate a new principle of fixed points, and we call it "*monotone principle of fixed points*".

A fixed point theorem for set-valued mappings in a complete metric space and some theorems on fixed points in arbitrary topological spaces are presented in this paper. Also, we describe a class of conditions sufficient for the existence of a fixed point which generalize several known results. We introduce the concept of a contraction principle and CS-convergence. With such an extension, a very general fixed point theorem is obtained to include a recent result of the author, which contains, as special cases, some results of J. Dugundji and A. Granas, F. Browder, D. W. Boyd and J. S. Wong, J. Caristi, T. L. Hicks and B. E. Rhoades, B. Fisher, W. Kirk and M. Krasnoselskij.

1. Introduction. Let X be a complete metric space with the metric ρ. In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle. Some of these generalizations refer to results containing the Schauder fixed point theorem. The purpose of the present paper is to consider a generalization of the contraction principle by introducing a "monotonicity" condition concerning the iterates of the mapping. We think that this condition may be adapted for other classes of mappings to obtain some extensions of known fixed point results.

In [**4**] Dugundji and Granas obtained a fixed point theorem which is a common generalization of results of Banach, Browder [**2**], Krasnoselskij [**8**], and many others. In this paper, we extend Dugundji and Granas's and Caristi's theorem and we describe a class of conditions sufficient for the existence of a fixed point which generalize several known results.

Let (X, ρ) be a metric space. A mapping $\theta\colon X \times X \to \mathbf{R}^0_+ := [0, +\infty)$, not necessarily continuous, is called *compactly positive* on X, if

$$\inf\{\theta(x, y)\colon \alpha \leqslant \rho(x, y) \leqslant \beta\} > 0$$

for each finite closed interval $[\alpha, \beta] \subset \mathbf{R}^0_+ \setminus \{0\}$. In a recent paper Dugundji and Granas [**4**] investigated a mapping T on a complete metric space (X, ρ) that satisfies the following condition: *there exist a compactly positive θ on X* such that

$$\text{(DG)} \qquad \rho[Tx, Ty] \leqslant \rho[x, y] - \theta(x, y) \quad \text{for all } x, y \in X,$$

and showed that such mappings have a fixed point in X.

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A mapping T: $X \to X$ satisfying (DG) is referred to as *weakly contractive*. For $x \in X$, $\sigma(x, \infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x. A function g mapping X into the reals is T-orbitally *lower semicontinuous* at p if $\{x_n\}$ is a sequence in $\sigma(x, \infty)$ and $x_n \to p$ implies that $g(p) \leqslant \liminf g(x_n)$. A space X is said to be T-orbitally complete iff every Cauchy sequence which is contained in $\sigma(x, \infty)$ for some $x \in X$ converges in X (cf. [**6** or **9**]).

2. Main result. In this section we introduce the concept of a "*monotone principle of fixed points*". We begin with a lemma which is fundamental.

LEMMA 1. *Let the mapping* γ: $\mathbf{R}_+ \to \mathbf{R}_+ := (0, +\infty)$ *have the properties*

$$(\gamma) \qquad (\forall t \in \mathbf{R}_+)\Big(\gamma(t) < t \wedge \limsup_{z \to t+0} \gamma(z) < t\Big).$$

If the bounded double sequence $(x_{m,n})$ *of real nonnegative numbers satisfies the inequality* $x_{m+1,n+1} \leqslant \gamma(x_{m,n})$, m, $n \in \mathbf{N}$, *then it converges to zero.*

PROOF. Since $(x_{m,n})$ is a bounded sequence in $\mathbf{R}_+$, there is a $t \geqslant 0$ such that $\limsup x_{m,n} = t$. We claim that $t = 0$. If $t > 0$, then

$$t = \limsup x_{m+1,n+1} \leqslant \limsup \gamma(x_{m,n}) \leqslant \limsup_{z \to t+0} \gamma(z) < t,$$

which is a contradiction. Consequently $t = 0$, $\lim x_{m,n} = 0$.

An immediate corollary of the preceding statement is

LEMMA 2 (TASKOVIĆ [**10**]). *Let the mapping* γ: $\mathbf{R}_+ \to \mathbf{R}_+$ *have the property* (γ). *If the sequence* (x_n) *of nonnegative real numbers satisfies the condition* $x_{n+1} \leqslant \gamma(x_n)$, $n \in \mathbf{N}$, *then the sequence* (x_n) *tends to zero. The velocity of this convergence is not necessarily geometrical.*

In connection with this, we shall introduce the concept of AT-condition in a space X; i.e., a metric space X which satisfies the condition of AT-type if for some $x \in X$ such that $A(T^n x, T^{n+1}x) \to 0$ $(n \to \infty)$ implies $\{A(T^n x, T^m x)\}$ is a bounded double sequence, where A: $X \times X \to \mathbf{R}^0_+$, $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous.

We are now in a position to formulate our main theorem.

THEOREM 1 (MONOTONE PRINCIPLE OF F.P.). *Let T be a mapping of a metric space (X, ρ) into itself and let X be T-orbitally complete with the condition of AT-type. Suppose that there exists a mapping* γ: $\mathbf{R}^0_+ \to \mathbf{R}^0_+$ *such that* (γ) *and*

$$(\mathrm{T}) \qquad A(Tx, Ty) \leqslant \gamma(A(x, y)) \quad \textit{for any } x, y \in X,$$

where A: $X \times X \to \mathbf{R}^0_+$, $x \mapsto A(x, Tx)$ *is T-orbitally lower semicontinuous* (*or A is continuous and* $A(x, x) = 0$) *and* $\rho[x, y] \leqslant A(x, y)$ *for all* x, $y \in X$. *Then T has a unique fixed point* $\xi \in X$ *and* $T^n x \to \xi$ *for each* $x \in X$.

PROOF. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}$ is a Cauchy sequence. Let n and m $(n < m)$ be any positive integers.

From (T) we have $A(T^n x, T^{n+1}x) \leqslant \gamma(A(T^{n-1}x, T^n x))$, $n \in \mathbf{N}$. Applying Lemma 2 to the sequence $\{A(T^n x, T^{n+1}x)\}$ we obtain $A(T^n x, T^{n+1}x) \to 0$ $(n \to \infty)$. This implies that $\{A(T^n x, T^m x)\}$ is a bounded sequence, since X satisfies the condition of AT-type. Also, from (T) we have

$$\rho[T^n x, T^m x] \leqslant A(T^n x, T^m x) \leqslant \gamma\big(A(T^{n-1}x, T^{m-1}x)\big).$$

Applying Lemma 1 to the sequence $\{A(T^n x, T^m x)\}$ we obtain $\lim A(T^n x, T^m x) = 0$; i.e., $\rho[T^n x, T^m x] \to 0$ $(m, n \to 0)$. This implies that $\{T^n x\}$ is a Cauchy sequence in X and, hence, by T-orbital completeness, there is a $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous at ξ,

$$\rho[\xi, T\xi] \leqslant A(\xi, T\xi) \leqslant \liminf A(T^n x, T^{n+1}x) = 0.$$

Thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}$ converges to a fixed point of T. (If A is continuous and $A(x, x) = 0$, then

$$\rho\big[T^{n+1}x, T\xi\big] \leqslant A\big(T^{n+1}x, T\xi\big) \leqslant \gamma\big(A(T^n x, \xi)\big) < A(T^n x, \xi)$$

implies $\rho[\xi, T\xi] \leqslant \lim A(T^n x, \xi) = A(\xi, \xi) = 0$; i.e., $T\xi = \xi$, and $\{T^n x\}$ converges to a fixed point of T.)

We complete the proof by showing that T can have at most one fixed point: for, if $\xi \neq \eta$ were two fixed points, then $0 < \rho[\xi, \eta] \leqslant A(\xi, \eta) = A(T\xi, T\eta) \leqslant \gamma(A(\xi, \eta)) < A(\xi, \eta)$, a contradiction. The proof is complete

Corollary 1 (J. Dugundji and A. Granas [4]). *Let (X, ρ) be a complete metric space, and $T\colon X \to X$ weakly contractive:*

$$\rho[Tx, Ty] \leqslant \rho[x, y] - \theta(x, y) \quad \textit{for all } x, y \in X,$$

where θ is compactly positive on X. Then T has a unique fixed point ξ, and $T^n x \to \xi$ for each $x \in X$.

Proof. Let $A(x, y) = \rho[x, y]$, $\gamma(t) = t - \theta(x, y)$ for $t \geqslant \theta(x, y)$ and $\gamma(t) = 0$, $0 \leqslant t < \theta(x, y)$. It is easy to see that A and γ satisfy all the required hypotheses in Theorem 1. By hypothesis, T is weakly contractive on X; therefore, X satisfies the condition of AT-type, i.e., ρT-type. Since completeness implies T-orbital completeness, it follows from the theorem that T has a fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

Corollary 2 (D. W. Boyd and J. S. Wong [1] and F. Browder [2]). *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists a continuous function Φ on $\mathbf{R}^0_+$ satisfying $\Phi(t) < t$ for $t > 0$ such that*

(BW) $$\rho[Tx, Ty] \leqslant \Phi(\rho[x, y]) \quad \textit{for all } x, y \in X.$$

Then T has a unique fixed point ξ and $\{T^n x\}$ converges to ξ for all x in X.

Proof. Let $A(x, y) = \rho[x, y]$ and $\gamma(t) = \Phi(t)$. It is easy to see that A and γ satisfy all the required hypotheses in Theorem 1. Also, from (BW), X satisfies the condition of AT-type, i.e., ρT-type.

COROLLARY 3. *Let T be a mapping of metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exist $\alpha \in [0,1)$ and*

$$A(Tx, Ty) \leqslant \alpha A(x, y) \quad \text{for all } x, y \in X,$$

where $A\colon X \times X \to \mathbf{R}^0_+$, $x \mapsto A(x, Tx)$ is lower semicontinuous and $\rho[x, y] \leqslant A(x, y)$ for all $x, y \in X$. Then T has a unique fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

3. Some localizations. Let X be a topological space, let $T\colon X \to X$, and let $B\colon X \to \mathbf{R}^0_+$ be a T-orbitally lower semicontinuous function on X.

In connection with this, we shall introduce the concept of TCS-convergence in a space X; i.e., a topological space X satisfies the condition of TCS-convergence if there exists a point $x \in X$ such that $B(T^n x) \to 0$ $(n \to \infty)$ implies $\{T^n x\}$ has a convergent subsequence.

THEOREM 2 (LOCALIZATION MONOTONE PRINCIPLE). *Let T be a mapping of a topological space X into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists a mapping $\gamma\colon \mathbf{R}^0_+ \to \mathbf{R}^0_+$ such that (γ) and*

$$\text{(LT)} \qquad B(Tx) \leqslant \gamma(B(x)) \quad \text{for all } x \in X,$$

where $B\colon X \to \mathbf{R}^0_+$ is T-orbitally lower semicontinuous and $B(x) = 0$ implies $T(x) = x$. Then T has a fixed point $\xi \in X$.

An immediate corollary of the preceding statement is

COROLLARY 4. *Let T be a mapping of a topological space X into itself with the property* (LT). *If for some $x \in X$ the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point $\xi \in X$.*

PROOF OF THEOREM 2. Let x be an arbitrary point in X and $\sigma(x, \infty)$ the orbit of x under T. Then, from (LT) we have

$$B(T^{n+1}x) \leqslant \gamma(B(T^n x)), \qquad n \in \mathbf{N}.$$

Applying Lemma 2 to the sequence $\{B(T^n x)\}$, we obtain $B(T^n x) \to 0$ $(n \to \infty)$. This implies (from TCS-convergence) that its sequence of iterates $\{T^n x\}$ contains a convergent subsequence $\{T^{n_i} x\}$ with limit $\xi \in X$. Since $B\colon X \to \mathbf{R}^0_+$ is T-orbitally lower semicontinuous,

$$B(\xi) \leqslant \liminf B(T^{n_i}(x)) = \liminf B(T^n x) = 0$$

implies that $B(\xi) = 0$, i.e., $T(\xi) = \xi$.

COROLLARY 5. *Let (X, ρ) be a complete metric space and $g\colon X \to X$ an arbitrary mapping. Suppose for all $x \in X$ that g satisfies*

$$\text{(LDG)} \qquad \rho[g(x), g^2(x)] \leqslant \rho[x, g(x)] - \theta(x, g(x)),$$

where θ is compactly positive on X. If $x \mapsto \rho[x, g(x)]$ is g-orbitally lower semicontinuous, then g has a fixed point in X.

PROOF. Let $B(x) = \rho[x, g(x)]$ which is lower semicontinuous on X, and let $\gamma(t) = t - \theta(x, g(x))$ for $t \geqslant \theta(x, g(x))$ and $\gamma(t) = 0$ for $0 \leqslant t < \theta(x, g(x))$; then B and γ satisfy all the required hypotheses in Theorem 2. (LDG) implies

$\rho[g^n x, g^{n+1}x] \to 0$ $(n \to \infty)$ and, since X is a complete metric space, we have (see the Lemma of Dugundji and Granas [**4**, p. 142]) that $\{g^n x\}$ converges to some $\xi \in X$, i.e., X that satisfies the condition of TCS-convergence. Hence, it follows from the theorem that g has a fixed point.

COROLLARY 6 (T. L. HICKS AND B. E. RHOADES [**8**]). *Let (X, ρ) be a complete metric space and T: $X \to X$ an arbitrary mapping. Suppose there exists an $x \in X$ such that*

$$\text{(HR)} \qquad \rho[Ty, T^2y] \leqslant h\rho[y, Ty], \qquad h \in [0,1),$$

for every $y \in \sigma(x, \infty)$. Then some $\xi \in X$ is a fixed point of T if $G(x) = \rho[x, T(x)]$ is T-orbitally lower semicontinuous.

PROOF. Let $B(x) = \rho[x, Tx]$ and $\gamma(t) = ht$. Since X satisfies the condition of TCS-convergence (X is a complete metric space and

$$\rho[T^n x, T^{n+k}x] \leqslant h^n(1-h)^{-1}\rho[x, Tx]),$$

applying Theorem 2 gives $T\xi = \xi$ for some $\xi \in X$.

COROLLARY 7 (J. CARISTI [**3**] AND W. A. KIRK [**7**]). *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists a lower semicontinuous function G of X into $\mathbf{R}^0_+$ such that*

$$\text{(CK)} \qquad \rho[x, Tx] \leqslant G(x) - G(Tx) \quad \textit{for all } x \in X.$$

Then T has a fixed point.

PROOF. Since X is complete and from (CK)

$$\sum_{i=0}^{\infty} \rho[x_i, x_{i+1}] < G(x_0),$$

X satisfies the condition of TCS-convergence. Letting $B(x) = G(x)$ and $\gamma(t) = t - \rho[x, Tx]$ for $t \geqslant \rho[x, Tx]$ and $\gamma(t) = 0$ for $0 \leqslant t < \rho[x, Tx]$ in (LT) gives

$$G(Tx) \leqslant G(x) - \rho[x, Tx], \qquad x \in X,$$

i.e., (CK). Hence, it follows from our theorem that T has a fixed point.

COROLLARY 8 (B. FISHER [**5**]). *If T is a mapping of the complete metric space X into itself satisfying the inequality*

$$\text{(FB)} \qquad \rho[T^2x, Ty] \leqslant \beta \max\{\rho[Tx, T^2x], \rho[y, Ty]\} \quad \textit{for all } x, y \textit{ in } X,$$

where $0 \leqslant \beta < 1$, then T has a unique fixed point.

PROOF. Let x be an arbitrary point in X. Then, for $y = x$, from (FB) we have

$$\rho[T^2x, Tx] \leqslant \beta \max\{\rho[T^2x, Tx], \rho[x, Tx]\} = \beta\rho[x, Tx].$$

Hence, for $B(x) = \rho[x, Tx]$, $\gamma(t) = \beta t$ $(\beta \in [0,1))$ and, since X satisfies the condition of TCS-convergence (X is a complete metric space and $\rho[T^n x, T^{n+k}x] \leqslant \beta^n(1-\beta)^{-1}\rho[x, Tx]$), applying Theorem 2 we obtain $T\xi = \xi$ for some $\xi \in X$. Uniqueness follows immediately from condition (FB).

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