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EXTREME POINTS IN DUALS OF COMPLEX OPERATOR SPACES

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ABSTRACT. We show that if X and Y are complex Banach spaces, and $K(X, Y)$ is the space of compact linear operators from X into Y, then $\operatorname{ext} B(K(X, Y)^*) = \operatorname{ext} B(X^{**}) \otimes \operatorname{ext} B(Y^*)$.

Introduction. We shall let X and Y be complex Banach spaces and we denote the space of compact linear operators from X into Y by $K(X, Y)$. The space of linear bounded operators is denoted $L(X, Y)$. $B(X)$ is the closed unit ball of X and X^* is the dual space of X. $\operatorname{ext} B(X)$ denotes the set of extreme points in $B(X)$.

For $x^{**} \in X^{**}$ and $y^* \in Y^*$, $x^{**} \otimes y^* \in K(X, Y)^*$ is defined by

$$x^{**} \otimes y^*(T) = x^{**}(T^* y^*).$$

In [3, Corollary 2], Fakhouri proved that for real spaces

$$\operatorname{ext} B(K(X, Y)^*) \subseteq \operatorname{ext} B(X^{**}) \otimes \operatorname{ext} B(Y^*).$$

A generalization of this result was given by Collins and Ruess in [2, Theorem 2.2]. The proofs given in [2 and 3] extend to complex spaces. Ruess and Stegall proved in [8, Theorem 1.3] the converse inclusion. Thus

$$(*) \qquad \operatorname{ext} B(K(X, Y)^*) = \operatorname{ext} B(X^{**}) \otimes \operatorname{ext} B(Y^*).$$

The proof in [8] is valid in the real case only. The object of this short note is to give a proof of the inclusion $\supseteq$ in $(*)$ for the complex case. The main result is

THEOREM 1. *For real and complex Banach spaces, we have*

$$(*) \qquad \operatorname{ext} B(K(X, Y)^*) = \operatorname{ext} B(X^{**}) \otimes \operatorname{ext} B(Y^*).$$

PROOF OF THE MAIN RESULT. We shall need Lemma 2 to test if $x^{**} \otimes y^*$ is an extreme point.

LEMMA 2. *Let $\Phi\colon X \to C(K)$ be a linear isometry and let $x_0^* \in B(X^*)$ with $\|x_0^*\| = 1$. Then $x_0^* \in \operatorname{ext} B(X^*)$ if and only if*

$$(\#) \qquad \begin{cases} K_0 = \{k \in K\colon \exists \lambda, |\lambda| = 1, \Phi^*(\partial_k) = \lambda x_0^*\} \neq \varnothing \text{ and if} \\ \|\mu\| = 1 = \|\Phi^*\mu\| \text{ with } \Phi^*\mu \in \operatorname{span}\{x_0^*\}, \text{ then} \\ \operatorname{support} \mu \subseteq K_0. \end{cases}$$

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PROOF. Assume (#) is satisfied. Suppose x_1^*, $x_2^* \in B(X^*)$ with $2x_0^* = x_1^* + x_2^*$. Choose $\mu_i \in C(K)^*$ such that $\|\mu_i\| = \|x_i^*\|$ and $x_i^* = \Phi^*\mu_i$ for $i = 1, 2$. Let $\nu = \mu_1 + \mu_2$. Then we have $1 = \|\nu\| = \|\Phi^*\nu\|$ and $\Phi^*\nu = x_0^*$. By (#), ν is supported by K_0. Since

$$1 = \|x_0^*\| \leqslant \|\nu\| = |\nu|(K_0) \leqslant \tfrac{1}{2}\big(|\mu_1| + |\mu_2|\big)(K_0) \leqslant 1,$$

μ_1 and μ_2 are supported by K_0 also. Hence $x_i^* = \Phi^*\mu_i = \lambda_i x_0^*$ for some $|\lambda_i| \leqslant 1$. This is clearly true if μ_i is a discrete measure. If μ_i is not discrete, then we use that we can approximate μ_i in the w^*-topology by discrete measures supported by K_0 and that Φ^* is w^*-continuous.

Thus

$$2x_0^* = x_1^* + x_2^* = \lambda_1 x_0^* + \lambda_2 x_0^*$$

with $|\lambda_i| \leqslant 1$ and it follows that $\lambda_1 = \lambda_2 = 1$. Hence x_0^* is an extreme point.

The converse implication is simple and we shall not give the details.

It is well known that in a compact convex set, the only probability measure representing an extreme point is the point measure with unit mass [**1**, Corollary I.2.4]. There can be many more complex representing measures but, as the next lemma shows, their support is on a small set.

LEMMA 3. *Assume X is a complex Banach space. Let $x_0^* \in \operatorname{ext} B(X^*)$ and let λ be a measure on $B(X^*)$ with $\|\lambda\| \leqslant 1$ such that*

$$x_0^*(x) = \int x^*(x)\, d\lambda(x^*) \quad \textit{for all } x \in X.$$

Then support $\mu \subseteq \{\alpha x_0^*\colon |\alpha| = 1\}$.

PROOF. There is a measurable function Φ on $B(X^*)$ such that $|\Phi| = 1$ a.e. $|\lambda|$ and $\lambda = \Phi|\lambda|$. Define $\omega\colon B(X^*) \to B(X^*)$ by $\omega(x^*) = \Phi(x^*)x^*$. ω is measurable. We define a new measure $\omega(|\lambda|)$ on $B(X^*)$ by

$$\omega(|\lambda|)(f) = \int f(\omega(x^*))\, d|\lambda|(x^*) = \int f(\Phi(x^*)x^*)\, d|\lambda|(x^*).$$

For $x \in X$, we get

$$\omega(|\lambda|)(x) = \int x(\Phi(x^*)x^*)\, d|\lambda|(x^*) = \int \Phi(x^*)x^*(x)\, d|\lambda|(x^*)$$

$$= \int x^*(x)\, d\lambda(x^*) = x_0^*(x).$$

Thus $\omega(|\lambda|)$ is a probability measure representing $x_0^* \in \operatorname{ext} B(X^*)$. By Corollary I.2.4 in [**1**], we get $\omega(|\lambda|) = \partial_{x_0^*}$. Suppose V is a compact subset in $B(X^*)$ with $|\lambda|(V) > 0$ and $V \cap \{\alpha x_0^*\colon |\alpha| = 1\} = \varnothing$. Let $\lambda_1 = \chi_V \lambda$ and $\lambda_2 = \lambda - \lambda_1$. Then we have $\lambda_1 \perp \lambda_2$ and $\|\lambda\| = \|\lambda_1\| + \|\lambda_2\|$. By Corollary 3 in [**7**], we get

$$\delta_{x_0^*} = \omega(|\lambda|) = \omega\big(|\lambda_1|\big) + \omega\big(|\lambda_2|\big).$$

If $f \geqslant 0$ is a continuous function on $B(X^*)$ with $f = 0$ on $\{\alpha x_0^*: |\alpha| = 1\}$ and $f = 1$ on $\{\alpha v: |\alpha| = 1, v \in V\}$, then we have

$$0 = \partial_{x_0^*}(f) \geqslant \omega(|\lambda_1|)(f)$$

$$= \int f(\Phi(x^*)x^*)\chi_V(x^*)\, d|\lambda|(x^*) \geqslant |\lambda|(V) > 0.$$

This contradiction shows that support $|\lambda| \subseteq \{\alpha x_0^*: |\alpha| = 1\}$.

Before we give the proof of Theorem 1, we shall give the complex version of Lemma 1.5 in **[8]**.

LEMMA 4. *Assume* $x_0^* \in \operatorname{ext} B(X^*)$ *and that* μ *is a positive measure on* $B(X^*)$ *with* $\|\mu\| \leqslant 1$. *If* $|x_0^*(x)| \leqslant \int |x^*(x)|\, d\mu(x^*)$ *for all* $x \in X$, *then* support $\mu \subseteq \{\alpha x_0^*: |\alpha| = 1\}$.

PROOF. As in the proof of Lemma 1.5 in **[8]**, we find $h \in L^\infty(B(X^*), \mu)$ such that

$$x_0^*(x) = \int x^*(x)h(x^*)\, d\mu(x^*) \quad \text{for all } x \in X.$$

We have $\|hd\mu\| \leqslant 1$ and by Lemma 3 it follows that $hd\mu$ has support in $\{\alpha x_0^*: |\alpha| = 1\}$. Since necessarily, $1 = \|\mu\| = \|hd\mu\|$, it follows that support $\mu \subseteq \{\alpha x_0^*: |\alpha| = 1\}$.

PROOF OF THEOREM 1. Let $x_0^{**} \in \operatorname{ext} B(X^{**})$ and $y_0^* \in \operatorname{ext} B(Y^*)$ and let $h^* = x_0^{**} \otimes y_0^* \in K(X, Y)^*$. Let $K = B(X^{**}) \times B(Y^*)$ with product w^*-topology. Let Φ: $K(X, Y) \to C(K)$ be defined by

$$\Phi(T)(x^{**}, y^*) = x^{**}(T^*y^*).$$

Φ is a linear isometry. Let $K_0 = \{k \in K: \exists \lambda, |\lambda| = 1, \Phi^*(\delta_k) = \lambda h^*\}$ and let $Z_0 = \{(\alpha x_0^{**}, \beta y_0^*): |\alpha| = |\beta| = 1\}$. Then clearly $Z_0 \subseteq K_0 \subseteq K$.

We shall use Lemma 2 to show that h^* is an extreme point. Suppose μ is a measure on K with $\|\mu\| = 1 = \|\Phi^*\mu\|$ and assume $\Phi^*\mu \in \operatorname{span}\{h^*\}$. Then $\Phi^*\mu = \lambda h^*$ and we may assume $\lambda = 1$.

Let $x^* \in X^*$ and $y \in Y$. Define $T \in K(X, Y)$ by $T(x) = x^*(x)y$. Then we get

$$x_0^{**}(x^*)y_0^*(y) = h^*(T) = \Phi^*\mu(T) = \mu(\Phi(T))$$

$$= \int_K \Phi(T)(x^{**}, y^*)\, d\mu(x^{**}, y^*)$$

$$= \int_K x^{**}(x^*)y^*(y)\, d\mu(x^{**}, y^*).$$

Let $\mu_1 = |\mu||_{B(X^{**})}$ and $\mu_2 = |\mu||_{B(Y^*)}$. Then we get

$$|x_0^{**}(x^*)y_0^*(y)| \leqslant \int_{B(X^{**})} |x^{**}(x^*)|d\mu_1(x^{**}).$$

Thus it follows that

$$|x_0^{**}(x^*)| \leqslant \int_{B(X^{**})} |x^{**}(x^*)|d\mu_1(x^{**}).$$

By Lemma 4, support $\mu_1 \subseteq \{\alpha x_0^{**}: |\alpha| = 1\}$. Similarly, support $\mu_2 \subseteq \{\beta y_0^{*}: |\beta| = 1\}$. Hence support $\mu \subseteq Z_0 \subseteq K_0$. By Lemma 2, h^* is an extreme point in $B(K(X, Y)^*)$.

Applications. Using Theorem 1, the proof of Theorem 5.6 in **[5]** extends to the complex case.

THEOREM 5. *Let X and Y be real or complex Banach spaces. If $K(X, Y)$ contains a proper M-summand, then Y contains a proper M-summand or X contains a proper L-summand.*

For reflexive spaces we get

COROLLARY 6. *Assume X is reflexive. If $K(X)$ contains a proper M-ideal, then X or X^* contains a proper M-summand.*

COROLLARY 7. *For $1 < p < \infty$, $K(l_p)$ contains no proper M-ideal.*

It is well known that the M-ideals coincide with the closed two-sided ideals in C^*-algebras. Thus Corollary 7 gives as a special case the well-known fact that $K(l_2)$ contains no proper closed two-sided ideals. Corollary 7 was first proved by Smith and Ward **[9]** and Flinn **[4]**. Corollary 6 extends their result to a much larger class of spaces.

From the proof of Theorem 5.7 in **[6]**, it follows that if $T \in L(X, Y)$ and $\varepsilon > 0$, then there exists $S \in L(X, Y)$, $y_0^* \in \operatorname{ext} B(Y^*)$ and $x_0^{**} \in \operatorname{ext} B(X^{**})$ such that $\|S - T\| < \varepsilon$ and $\|S\| = x_0^{**}(S^* y_0^*)$. For $h = x_0^{**} \otimes y_0^* \in \operatorname{ext} B(K(X, Y)^*)$, let $\hat{h}$ denote the natural extension to $L(X, Y)$ defined by $\hat{h}(T) = x_0^{**}(T^* y_0^*)$. Thus we have that the convex hull of $\{\hat{h}: h \in \operatorname{ext} B(K(X, Y)^*)\}$ is w^*-dense in $B(L(X, Y)^*)$.

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