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ANALYTICITY IN THE BOUNDARY OF A PSEUDOCONVEX DOMAIN

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ABSTRACT. Let D be a bounded pseudoconvex domain with C^∞ boundary in $\mathbf{C}^n$, $A^\infty(D)$ the algebra of functions holomorphic in D and C^∞ up to the boundary, and M a compact real-analytic manifold in the boundary which is integral for the complex structure of the boundary and which has no complex tangent vectors. A necessary and sufficient condition that each element of $A^\infty(D)$ be real-analytic on M is that the germ of the complexification of M be in the boundary. Examples indicate that the quasi-analyticity of $A^\infty(D)$ along M is possible even in the absence of complex manifolds in the boundary.

1. Introduction. We call a smooth manifold M in the boundary of a domain an *integral manifold* if its tangent space at each point is contained in the maximal complex subspace of the tangent space of the boundary. M is *totally real* if it has no complex tangent vectors; more precisely, if J is the almost complex structure, the condition is that $T_p(M) \cap JT_p(M) = 0$ for all $p \in M$. A well-known theorem due to Stein states that holomorphic functions which are Lipschitz on $\bar{D}$ are twice as smooth when restricted to integral curves. (For the precise statement we refer the reader to [**9**, Corollary 2, p. 443].) In this note we investigate what conditions on D (or ∂D) imply high regularity of functions in $A^\infty(D)|M = \{f|M;\ f \in A^\infty(D)\}$; here M is a compact totally real real-analytic integral manifold in ∂D. Our results depend on the notion of a complexification of such a manifold. Suppose M has real dimension m. Locally (near $p \in M$) we take a real-analytic parametrization $\phi\colon V \to M$, where V is a neighborhood of 0 in $\mathbf{R}^m$ and $\phi(0) = p$. The holomorphic extension Φ of ϕ to a neighborhood V' of 0 in $\mathbf{C}^m$ is nonsingular since M is totally real; then $\Phi(V')$ is a complexification of M near p. Using the compactness of M we combine these to get a complex submanifold M' of a neighborhood W of M which has complex dimension m and which contains M as a submanifold. Details of this construction are in [**10**, p. 1274]. Note that, assuming the connectedness of $M' \cap W$, for each real-analytic function on M there are a neighborhood W' of M and a unique extension of the function to $H(W' \cap M')$. (Here, as elsewhere, $H(N)$ denotes the algebra of holomorphic functions on the (connected) complex manifold N.) Our main result can then be stated as follows.

THEOREM. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^∞ boundary, M a compact totally real real-analytic integral manifold in ∂D, and M' a complexification of M in W. Then each element of $A^\infty(D)|M$ is real-analytic if and only if there is a neighborhood $U \subseteq W$ of M so that $U \cap M' \subseteq \partial D$.*

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The proof of this theorem is in §2. We remark that obviously pseudoconvexity is required in the theorem; furthermore, some minimal smoothness of the boundary is necessary. In fact, Sibony constructed in [**8**, p. 973] a bounded pseudoconvex domain in $\mathbf{C}^2$ (with nonsmooth boundary) so that all bounded holomorphic functions on the domain extend to be holomorphic on a strictly larger domain.

Motivation for this work came from a study of interpolation in [**6**]; there an example is given of a class of domains for which $A^\infty(D)$ gains a good deal of smoothness upon restriction to an integral curve. In §3 we further discuss this example as a contrast to the theorem above. In particular, we give the following

EXAMPLE. There exists a convex domain $D \Subset \mathbf{C}^2$ which is strongly pseudoconvex off of a line segment K so that $A^\infty(D)$ is quasi-analytic along a subinterval of K.

Our proof of the theorem depends on the identification of the spectrum of the algebra A^∞ given by Hakim and Sibony in [**3**, Theorem 1, p. 128]. Recall that $A^\infty(D)$ is a Fréchet algebra with the family of norms given by

$$P_N(f) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\|D^\alpha f\|_{\bar{D}};$$

here, as elsewhere in this note, $\|g\|_X$ denotes the supremum of $|g|$ on X.

THEOREM (HAKIM-SIBONY). *If D is a bounded pseudoconvex domain with C^∞ boundary, then the space of nonzero continuous complex homomorphisms of $A^\infty(D)$ can be identified with $\bar{D}$.*

2. Proof of the theorem. Suppose that, for some neighborhood U of M, $U \cap M' \subseteq \partial D$. If $f \in A^\infty(D)$, then $\bar{\partial} f \equiv 0$ in $\bar{D}$, so f is holomorphic on $U \cap M'$. It follows that f is real-analytic on M. Thus each element of $A^\infty(D)|M$ is real-analytic.

For the nontrivial part of the proof, we assume each element of $A^\infty(D)|M$ is real-analytic and fix a point $p \in M$. For each $f \in A^\infty(D)|M$ there is a neighborhood V of p (depending on f) so that f extends to be holomorphic on $V \cap M'$. Our first step is to remove the apparent dependence of V on f (cf. the argument in [**3**, p. 131]). Let $B(r)$ denote the open ball with center p and radius $r > 0$; let $X(r)$ be the Fréchet space of pairs (F, f) with $F \in H(B(r) \cap M')$, $f \in A^\infty(D)$, and $F = f$ on $B(r) \cap M$; and, let $\rho(r) : X(r) \to A^\infty(D)$ be the restriction map. We know that the union of the images of $\rho(r)$ over $1/r = 1, 2, 3, \ldots$ is $A^\infty(D)$, so, for some r_1, the image of $\rho(r_1)$ is of the second category in $A^\infty(D)$. By the open mapping theorem for Fréchet spaces (e.g., [**7**, p. 47]), $\rho(r_1)$ is surjective. Thus, if $V = B(r_1)$, each element of $A^\infty(D)|M$ extends to be holomorphic on $V \cap M'$.

The second step is to show that $V \cap M' \subseteq \bar{D}$. Fix a point $q \in V \cap M'$ and define a complex homomorphism $\chi : A^\infty(D) \to \mathbf{C}$ by $\chi(f) := F(q)$ if $f \in A^\infty(D)$ and F is an extension of f which is holomorphic on $V \cap M'$. Since the extension is unique, χ is well defined, and the following argument shows that χ is continuous: If $g \in A^\infty(D)$, then $|\chi(g)| \le \|g\|_{\bar{D}}$, for otherwise $g - \chi(g)$ would be invertible in $A^\infty(D)$, an impossibility. Thus, if $f_j \to f$ in $A^\infty(D)$, from $\|f_j - f\|_D \to 0$ it follows that $\chi(f_j) \to \chi(f)$. Hence, χ is continuous. By the aforementioned result of Hakim and Sibony, χ is given by evaluation at a point of $\bar{D}$, and it is clear that this point must be q. If follows that $q \in \bar{D}$ and so $V \cap M' \subseteq \bar{D}$.

The third step is to show that, in fact, $V \cap M' \subseteq \partial D$. For this we use the fact that there is a function $\sigma \in C(\bar{D})$ which is plurisubharmonic on D and satisfies $\sigma < 0$ on D while $\sigma = 0$ on ∂D; this is a simple form of the theorem of Diederich and Fornaess [**2**, Theorem 1, p. 131] on bounded plurisubharmonic exhaustion functions. We claim that σ is actually plurisubharmonic on $V \cap M'$. To see this, fix $q \in V \cap M'$, let $\mathbf{n}$ be the outward unit normal to ∂D at q, and let $V' \subset V$ be a small neighborhood of q. Since $V \cap M' \subseteq \bar{D}$, if $\varepsilon > 0$ is small, then

$$\{t - \varepsilon\mathbf{n};\ t \in V' \cap M'\} \subseteq D.$$

Thus $\sigma(t)$ is the uniform limit on $V' \cap M'$ of the plurisubharmonic functions $\sigma_\varepsilon(t) := \sigma(t - \varepsilon\mathbf{n})$ as $\varepsilon \to 0$; it follows that σ is plurisubharmonic on $V' \cap M'$. Since q was arbitrary, σ is plurisubharmonic on $V \cap M'$, giving the claim. Now σ attains its maximum value at the (relative) interior point p of $V \cap M'$; by the maximum principle, $\sigma \equiv 0$ on $V \cap M'$. Thus $V \cap M' \subseteq \partial D$.

We have shown that, for each $p \in M$, there exists a neighborhood V of p so that $V \cap M' \subseteq \partial D$. It follows that there is a neighborhood $U \subseteq W$ of M so that $U \cap M' \subseteq \partial D$.

REMARK. If $A(D) := H(D) \cap C(\bar{D})$, then it is easy to see that the assumption that $U \cap M' \subseteq \partial D$ for a neighborhood U of M implies that each element of $A(D)|M$ is real-analytic. In fact, fixing $f \in A(D)$ and $q \in U \cap M'$, we get that f is locally near q the uniform limit on M' of holomorphic functions by arguing as for σ in step 3 above. It follows that $f|M$ is real-analytic.

3. Example of quasi-analyticity in the boundary. For the example we choose two nonnegative even functions ϕ and χ in $C^\infty(\mathbf{R})$ so that

(a) each is strictly convex off its zero set;
(b) $\chi^{-1}(0) = [-2, 2]$;
(c) $\phi^{-1}(0) = \{0\}$; and
(d) ϕ vanishes to infinite order at 0.

From [**6**, Example 4.1] we recall the domain D, defined near $K := [-2, 2] \times \{0\}$ in $\mathbf{C}^2$, by

$$D := \left\{(z, w); u + \chi(x) + \phi(y) + v^2\left(1 + \frac{1}{100}|z|^2\right) < 0\right\};$$

here we use the notation $z = x + iy$, $w = u + iv$. D is convex and strongly pseudoconvex off of K, and K is an integral curve. We put $L := [-1, 1] \times \{0\}$ and

$$I_k = I_k(\phi) := \int_0^1 \phi'(t)t^{-k}\,dt \quad \text{for } k \geq 1.$$

LEMMA 1. *Given $f \in A^\infty(D)$ there exists $C > 0$ so that*

$$\|\partial^k f/\partial x^k\|_L \leq Ck!I_k \quad \textit{for } k \geq 1.$$

PROOF. Lemma 4.1 of [**6**] gives this estimate with L replaced by $\{(0, 0)\}$, and one only needs to check that the estimate holds uniformly on L. For the convenience of the reader, we sketch the proof. If $k \geq 1$ then

$$\frac{\partial^k f}{\partial x^k}(a, 0) = -\int_0^1 \frac{d}{dt}\left[\frac{\partial^k f}{\partial x^k}(a, -\phi(t))\right]\,dt + \frac{\partial^k f}{\partial x^k}(a, -\phi(1))$$

whenever $-1 \leq a \leq 1$. The integrand is bounded above by $k!\|\partial f/\partial w\|_{\bar{D}}\phi'(t)t^{-k}$ because of the Cauchy estimates for $\partial f/\partial w$ on discs in $\bar{D}$ of the form

$$\{z;\ |z-a| \leq t\} \times \{-\phi(t)\};$$

the second term is similarly bounded above by $k!\|f\|_{\bar{D}}$. This gives the desired estimate.

The lemma shows that we can get good regularity for $A^\infty(D)|L$ by choosing ϕ so that $I_k(\phi)$ grows slowly with k. The proof of the main theorem shows that we cannot choose ϕ so that, for some $C_1 > 0$,

$$(*) \qquad I_k(\phi) \leq C_1^k \quad \text{for } k \geq 1.$$

Here is a more direct proof of this: Put

$$\psi(t) := \begin{cases} 0 & \text{if } t < 1, \\ \phi'(1/t) & \text{if } t \geq 1. \end{cases}$$

The holomorphic Fourier transform F of ψ defined by

$$F(z) := \int_{-\infty}^{\infty} \psi(t)e^{itz}\,dt \qquad (z \in \mathbf{C})$$

would, if $(*)$ held, be an entire function of exponential type (a simple estimate); by the Paley-Wiener Theorem, F would be the Fourier transform of a function with compact support, so ψ would have compact support. Thus $(*)$ implies $\phi \equiv 0$ near 0, contradicting (c) above. In the following lemma we indicate one possible construction of a ϕ whose growth rate approximates $(*)$.

LEMMA 2. *Suppose $\{a_k\}$ is an unbounded increasing sequence with $a_1 \geq 1$. Then there exists a function ϕ of the required form with*

$$I_k(\phi) \leq a_k^k \quad \textit{for } k \geq 1.$$

PROOF. Fix $\lambda \in C^\infty(\mathbf{R})$ so that $0 \leq \lambda \leq 1, \lambda(t) \equiv 0$ if $t \leq 1$, and $\lambda(t) \equiv 1$ if $t \geq 2$. If $j \geq 1$, let $c_j := \max\{\|a_j^k\lambda^{(k)}(a_j t)\|_{\mathbf{R}};\ 0 \leq k \leq j\}$; then $1 \leq c_j < \infty$. We define

$$\psi(t) := \sum_{j=1}^{\infty} \lambda(a_j t)t^j/(c_j j^j) \quad \text{for } t \geq 0.$$

Then ψ is infinitely differentiable, and $\psi > 0$ if $t > 0$. A rather crude estimate gives that, for $k \geq 2$,

$$\int_0^1 \psi(t)t^{-k}\,dt = \sum_{j=1}^{\infty} \int_{1/a_j}^{1} \lambda(a_j t)t^{j-k}/(c_j j^j)\,dt$$
$$\leq (k-1)a_k^k + 1.$$

If we choose ϕ to be even and satisfy $\phi(0) = \phi'(0) = 0$ while $\phi''(t) = \psi(t)$ for $t \geq 0$, then integration by parts gives that, for some $C_1 > 0$, $I_k(\phi) \leq C_1 a_k^k$ for $k \geq 1$. Dividing ϕ by C_1 gives the desired result.

EXAMPLE. Let $a_k = \log k$ for $k \geq 3$, and let ϕ be the corresponding function given in Lemma 2. By Lemma 1, if $f \in A^\infty(D)$, then there exists $C > 0$ so that

$$\|\partial^k f/\partial x^k\|_L \leq C(k \log k)^k \quad \text{for } k \geq 3.$$

Since $\sum 1/(k \log k) = \infty$, the Denjoy-Carleman Theorem (e.g., [**4**, Chapter IV, pp. 101 ff.]) implies that $A^\infty(D)|L$ is quasi-analytic. We remark that with the choice $\chi(2+t) = \phi(t)$ (for $t \geq 0$) it is straightforward to check that $A^\infty(D)|K$ is quasi-analytic.

The above example gives a result about peak sets for $A^\infty(D)$. Recall that a closed set E in ∂D is a *peak set* for $A^\infty(D)$ if there exists a function $g \in A^\infty(D)$ with $g = 0$ on E while $\operatorname{Re} g > 0$ on $\bar{D} \backslash E$. K is a peak set for $A^\infty(D)$ (take $g = -w$), but no subset E of $(-1,1) \times \{0\}$ is a peak set for $A^\infty(D)$. In fact, if such a set E were a peak set with corresponding function g, the function $f = \exp\left(-1/\sqrt{g}\right) \in A^\infty(D)$ would vanish to infinite order on E. By the quasi-analyticity of $A^\infty(D)|L$, $f \equiv 0$ on L, so $E \supseteq L$, a contradiction. (A different proof of a related fact about peak sets in K is given in [**5**, Example 1.1].)

REMARK. In contrast to the above phenomena, $A(D)$ gains no regularity upon restriction to K in the above examples. More precisely, $A(D)|K = C(K)$, i.e., K is an interpolation set for $A(D)$. The proof is as follows: Since K is a peak set, a well-known result from the theory of uniform algebras (e.g., [**1**, Corollary 2.4.3, p. 104]) implies that $A(D)|K$ is uniformly closed in $C(K)$. In addition, the Stone-Weierstrass Theorem implies that holomorphic polynomials are dense in $C(K)$. Thus $A(D)|K = C(K)$.

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