PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 3, July 1985

ERGODIC ACTIONS OF THE MAPPING CLASS GROUP

HOWARD MASUR[1]

ABSTRACT. We prove that the horocyclic flow on the moduli space of a compact Riemann surface is ergodic. We also show that the mapping class group acts ergodically on Thurston's space of measured foliations.

1. Introduction. The Teichmüller extremal maps define a geodesic flow on Teichmüller space which projects to a flow on the moduli space. The Teichmüller discs allow one to define a horocyclic flow which also projects. In a previous paper [**6**] we showed the geodesic flow is ergodic. By identifying geodesics in Teichmüller space with pairs of projective measured foliations we showed the mapping class group acts ergodically on the product of the space of projectivized foliations with itself. The motivation for this paper was the question of whether the action is ergodic on the full (unprojectivized) space of foliations. This is equivalent to asking whether there are nontrivial sets of invariant horospheres.

We are able to show there is no measure theoretically nontrivial set of invariant horocycles which implies the same statement about horospheres. To prove the ergodicity of the horocyclic flow we use the ergodicity of the geodesic flow and methods developed by Hedlund in 1939 to study flows on surfaces of constant negative curvature. Hedlund's ideas have in fact led to a general phenomenon. If there is an $\mathrm{SL}(2,\mathbf{R})$ action on a space then the geodesic action $\left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]$ is ergodic if and only if the horocycle action $\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ is. This is precisely the set-up in this paper where we adapt Hedlund's original estimates to our situation.

2. Preliminaries. We recall the basic notions from the theory of measured foliations and Teichmüller spaces needed in this paper. For greater detail the reader may consult [**1**, **2** or **4**].

A measured foliation F on C^∞ surface M with singularities at $p_1, \ldots, p_n$ with orders $k_1, \ldots, k_n$ is given by

(i) An open cover of $M - \{p_1, \ldots, p_n\}$ by simply connected open sets U_i.

(ii) Closed C^∞ 1-forms ϕ_i on U_i satisfying $\phi_i = \pm\phi_j$ on $U_i \cap U_j$.

(iii) Local charts $z_j \colon V_j \to \mathbf{R}^2$ near each p_j such that, in $U_i \cap V_j$, $\phi_i = \operatorname{Im} z_j^{k_j/2}\, dz_j$ for some branch of $z_j^{k_j/2}$.

The leaves of F are the integral curves of the vector field along which ϕ_i vanishes. For any rectifiable curve γ, $i(F,\gamma)$ denotes the transverse length of γ computed in each local chart by $\int_\gamma |\phi_i|$. (ii) indicates this is well defined. For $\{\beta\}$ a homotopy class of simple closed curves, define $i(F,\{\beta\})$ or simply $i(F,\beta)$ as $\inf_{\beta'\sim\beta} i(F,\beta')$.

Received by the editors March 6, 1984 and, in revised form, September 28, 1984.
1980 *Mathematics Subject Classification*. Primary 32G15; Secondary 28D05, 30F30.
Key words and phrases. Measured foliations, horocyclic flow.
[1]Supported in part by NSF Grant MCS 8201310.

0002-9939/85 $1.00 + $.25 per page

We say F_1 is *equivalent* to F_2 if $i(F_1, \beta) = i(F_2, \beta)$ for all classes β and F_1 is *projectively equivalent* to F_2 if F_1 is equivalent to λF_2 for some $\lambda > 0$. If F is defined by 1-forms ϕ_i, λF is defined by $\lambda \phi_i$.

The space of equivalence classes is denoted MF. The space of projective classes is PF. MF is given the weak or product topology: $\lim_{n\to\infty} F_n = F_0$ if $\lim_{n\to\infty} i(F_n, \beta) = i(F_0, \beta)$ for all β. PF is given the induced topology.

A fundamental result of Thurston's is

$$MF \cong \mathbf{R}^{6g-6} - \{0\}, \qquad PF \cong S^{6g-7}.$$

Let $\mathrm{Mod}(g) = \mathrm{Diff}_+(M)/\mathrm{Diff}_0(M)$ be the mapping class group. It acts on MF as follows. For any $f \in \mathrm{Mod}(g)$ and $F \in MF$, $f \cdot F$ is the foliation with singularities $f(p_1), \ldots, f(p_n)$ with orders $k_1, \ldots, k_n$, and 1-forms $(f^{-1})^* \phi_i$ defined on $f(U_i)$. It is easy to check that $f \cdot F$ is defined on equivalence classes of both f and F and also defines an action on PF.

MF has a measure invariant under the action of $\mathrm{Mod}(g)$, constructed in [**6**]. Locally it is described as follows. For any F_0 with simple singularities (all $k_i = 1$) let $\tilde{M}$ be the double cover of M branched over the singularities of F such that the ϕ_i lift to a global 1-form $\tilde{\phi}$ on $\tilde{M}$. There is a natural involution τ on $\tilde{M}$ such that $\tau^* \tilde{\phi} = -\tilde{\phi}$. Let $\gamma_1, \ldots, \gamma_{6g-6}$ be a basis for the homology on M odd with respect to τ. The map

$$F \to \left(\int_{\gamma_1} \tilde{\phi}, \ldots, \int_{\gamma_{6g-g}} \tilde{\phi} \right)$$

is a local homeomorphism near F_0. The measure m on MF is Lebesgue measure induced by this homeomorphism.

Next let X be a compact Riemann surface of genus $g \geq 2$. A holomorphic quadratic differential q assigns to each holomorphic local coordinate z, a holomorphic function $q(z)$ such that $q(z)\,dz^2$ is invariant under change of coordinates. Then q defines a pair of transverse measured foliations F_q and F_{-q} defined locally by $\operatorname{Im} q^{1/2}\,dz$ and $\operatorname{Re} q^{1/2}\,dz$, resp. These are called the horizontal and vertical measured foliations of q.

Conversely, if F_1 and F_2 are transverse they define a Riemann surface X and quadratic differential q such that $F_q = F_1$, $F_{-q} = F_2$. Namely, find local coordinates (x, y) away from the singularities such that F_1 is given by dy, F_2 by dx. Then $z = x + iy$ are holomorphic coordinates and $q = dz^2$. Thus we can identify the set $Q \subset MF \times MF$ of pairs of transverse measured foliations with the space of nonzero holomorphic quadratic differentials on compact Riemann surfaces. This is the cotangent bundle $Q \to T_g$ where T_g is the Teichmüller space.

Let $Q_0 = \{q \in Q\colon \int |q| = 1\}$. Q carries the product measure on $MF \times MF$ and Q_0 the induced measure denoted μ on the unit sphere. It was shown in [**6**] that $MF \times MF - Q$ has zero measure.

We define a flow on Q_0 known as the Teichmüller flow. For any $q \in Q_0$ given by the pair F_q, F_{-q}, and $t \in \mathbf{R}$, let $G_t(q)$ be given by the pair $e^{-t} F_q$, $e^t F_{-q}$.

From the point of view of Teichmüller's theory $G_t(q_t)$ is the terminal quadratic differential under the Teichmüller map with dilation $k\bar{q}/|q|$,

$$t = \frac{1}{2} \log \frac{1+k}{1-k}.$$

The modular group preserves the flow.

THEOREM [**6**]. *$Q_0/\mathrm{Mod}(g)$ has finite measure. The flow on $Q_0/\mathrm{Mod}(g)$ is ergodic.* $\mathrm{Mod}(g)$ *acts ergodically on $PF \times PF$.*

Q_0 not only has a geodesic flow, but a horocyclic flow as well. This was originally discussed in [**5**]. We recall the definition.

Given $q \in Q_0$ on the surface X, the Teichmüller disc D_q consists of points (f_z, X_z) in T_g where $f_z\colon X \to X_z$ is the Teichmüller map with dilation $z\overline{q}/|q|$, $|z| < 1$. This defines a map

$$f\colon D = \{z\colon |z| < 1\} \to T_g$$

which is an isometry from the Poincaré metric on the unit disc to the Teichmüller metric. A horocycle in D_q is the image of a horocycle in D. In particular, this defines a right horocyclic flow on Q_0. Take the horocycle through X for which the Beltrami coefficient $\overline{q}/|q|$ is the inward normal tangent vector. This vector is $df(o)(v)$ where v is a positive real vector. Then for each $t \in \mathbf{R}$, move along the horocycle clockwise hyperbolic distance t if $t > 0$, counterclockwise distance $|t|$ if $t < 0$. The inward normal at that point is the Beltrami coefficient $\overline{q}_t/|q_t|$ for some $q_t \in Q_0$. The flow is

$$C_R(q, t) = q_t.$$

Note. In [**3**] the horocyclic flow is defined by using tangent rather than normal vectors.

The modular group also preserves the horocyclic flow.

3. Proof of the theorems.

THEOREM 1. *The horocyclic flow on $Q_0/\mathrm{Mod}(g)$ is ergodic.*

For any $\theta \in \mathbf{R}$ and $V \subset Q_0$ let $V_\theta = \{e^{i\theta} q\colon q \in V\}$.

LEMMA 1. $\mu(V_\theta) = \mu(V)$.

PROOF. The computation is local. Let V be a simply connected set in Q_0 consisting of q with simple zeroes. Each q may be lifted to a holomorphic 1-form $\tilde{q}$ on $\tilde{M}$. Pick a basis $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{6q-6}$ for $H_1(\tilde{M}, Z)^-$. Then the local coordinates

$$x_i = \int_{\tilde{\gamma}_i} \operatorname{Re} \tilde{q}\, dz, \quad y_i = \int_{\tilde{\gamma}_i} \operatorname{Im} \tilde{q}\, dz, \qquad 1 \le i \le 6g-6,$$

define the measure μ. Then the measure on $e^{i\theta}V$ is defined by the local coordinates

$$\cos\frac{\theta}{2} x_i - \sin\frac{\theta}{2} y_i, \quad \sin\frac{\theta}{2} x_i + \cos\frac{\theta}{2} y_i, \qquad i = 1, \ldots, 6g-6.$$

The transformation

$$(x_i, y_i) \to \left(\cos\frac{\theta}{2} x_i - \sin\frac{\theta}{2} y_i, \sin\frac{\theta}{2} x_i + \cos\frac{\theta}{2} y_i\right)$$

is measure preserving.

Now for any two points X, Y on a horocycle Y clockwise from X, $\overline{s}$ apart on the horocycle, let s be their distance apart on the geodesic. Then s depends only on $\overline{s}$ not on the two points, the function $s = h(\overline{s})$ is 1-1 and

$$\lim_{\overline{s} \to \infty} h(\overline{s}) = \infty.$$

The following lemma is standard [**3**].

LEMMA 2. *Suppose l is the geodesic from X to Y with tangent vector v at X pointing toward Y and vector v_s at Y pointing away from X. Then the angle θ v makes with the normal approaches* 0 *as $s \to \infty$, and v_s makes an angle $\pi - \theta$ with the inward normal.*

What this lemma says is that the image of a set under the horocyclic flow followed by a suitable rotation is arbitrarily close to the image under the geodesic flow for large $\overline{s}$ and s, resp. This should make the estimates in the theorem plausible.

PROOF OF THEOREM 1. We are able to follow Hedlund's proof [**3**] translated to this context. Suppose $V \subset Q_0/\mathrm{Mod}(g)$ is an invariant set under the horocyclic flow and $\mu(V) > 0$. We wish to show $\mu(V \cap W) > 0$, for any W with $\mu(W) > 0$.

Since the geodesic flow is ergodic,

$$\lim_{s-t\to\infty} \frac{1}{s-t} \int_t^s \mu(G_\tau(V) \cap W_\pi)\, d\tau = \frac{\mu(V)\mu(W_\pi)}{\mu(Q_0/\mathrm{Mod}(g))} = m_0.$$

This implies there is a sequence $s_n \to \infty$ with

$$\mu(G_{s_n}(V) \cap W_\pi) \geq m_0/2. \tag{1}$$

Now $V = V_\theta \cap V \cup (V - V_\theta \cap V)$ for any θ so

$$G_{s_n}(V) \cap W_\pi = G_{s_n}(V_\theta \cap V) \cap W_\pi \cup (G_{s_n}(V - V_\theta \cap V) \cap W_\pi).$$

Now $\mu(V - V_\theta \cap V) \to 0$ as $\theta \to 0$ so

$$\mu(V - V_\theta \cap V) < m_0/4, \qquad \theta \text{ small enough.} \tag{2}$$

Then

$$\mu(G_{s_n}(V \cap V_\theta) \cap W_\pi) > m_0/4 \tag{3}$$

by (1), (2) and the fact that the geodesic flow preserves measure. Thus,

$$\mu(G_{s_n}(V_\theta) \cap W_\pi) > m_0/4, \qquad \theta \text{ small enough.} \tag{4}$$

Again for θ small enough, $\mu(W_\pi - W_{\pi-\theta} \cap W_\pi) < m_0/8$. Thus, by (4),

$$\mu(G_{s_n}(V_\theta) \cap W_{\pi-\theta}) \geq \mu(G_{s_n}(V_\theta) \cap W_{\pi-\theta} \cap W_\pi) \geq m_0/8. \tag{5}$$

Now V is assumed invariant under the horocyclic flow so

$$V = C_R(V, \overline{s}_n) = (G_{s_n}(V_{\theta_n}))_{-(\pi-\theta_n)}$$

by Lemma 2 where $\lim_{n\to\infty} \theta_n = 0$ and $s_n = h(\overline{s}_n)$. Intersecting with W we have

$$\begin{aligned} \mu(V \cap W) &= \mu((G_{s_n}(V_{\theta_n}) \cap W_{\pi-\theta_n})_{-\pi+\theta_n}) \\ &= \mu(G_{s_n}(V_{\theta_n}) \cap W_{\pi-\theta_n}) \geq m_0/8 > 0 \end{aligned}$$

for n large enough by (5). This proves the flow is ergodic.

THEOREM 2. $\mathrm{Mod}(g)$ *acts ergodically on* MF.

PROOF. To any $F \in MF$ let $H(F) = \{q \in Q_0 \colon F_{-q} = F\}$. We call $H(F)$ the horosphere based at F. In [**5**] we showed that, for uniquely ergodic F, $H(F)$ has the "stable manifold" property that for two points $q^1, q^2 \in H(F)$, $G_t(q^1)$, $G_t(q^2)$ are quadratic differentials on surfaces whose Teichmüller distance apart approaches zero as $t \to \infty$. Now by Lemma 3.1 of [**5**], the horocycle determined by q is contained in the horosphere through q. By Theorem 1, any $\mathrm{Mod}(g)$ invariant set of horocycles has measure zero or its complement does. The same must be true then of invariant sets of horospheres.

Now suppose $E \subset MF$ is $\mathrm{Mod}(g)$ invariant and $H(E)$ the corresponding set of horospheres. Suppose E and E^c both have nonzero measure. Then so do $(E \times MF) \cap Q$ and $(E^c \times MF) \cap Q$ and so do their projection to Q_0 and they are invariant as well. But the latter sets are $H(E)$ and $H(E^c)$ respectively, and we have a contradiction.

REFERENCES

1. L. Bers, *Finite dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), 131–172.
2. A. Fathi, F. Laudenbach and V. Poenaru et al., *Travaux de Thurston sur les surfaces*, Asterisque **66–67** (1979).
3. G. Hedlund, *Fuchsian groups and mixtures*, Ann. of Math. (2) **40** (1939), 370–383.
4. A. Marden, *Geometric relations between homeomorphic Riemann surfaces*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 1001-1017.
5. H. Masur, *Transitivity properties of the horocyclic and geodesic flows on moduli space*, J. Analyse Math. **39** (1981), 1-10.
6. ——, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), 169–200.
7. W. Thurston, *On the dynamics of diffeomorphisms of surfaces*, Preprint.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680