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TOTALLY REAL EMBEDDINGS OF S^3 IN $\mathbf{C}^3$

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ABSTRACT. An explicit totally real embedding of S^3 in $\mathbf{C}^3$ is exhibited. It has the form $F(z,w) = (z,w,P(z,w))$ where P is a (nonholomorphic) polynomial of degree 4, and (z,w) ranges over the unit sphere in $\mathbf{C}^2$.

Suppose M is a smooth manifold and $F\colon M \to \mathbf{C}^n$ is an embedding. We say that F is a *totally real* embedding if, for every point of $F(M)$, the tangent space to $F(M)$ contains no nonzero complex subspace. In [**1**] Gromov stated that there exist totally real embeddings of the 3-sphere S^3 into $\mathbf{C}^3$, but no details are given there. In [**3**] Stout and Zame raise the problem of how to construct such embeddings.

We regard S^3 as the unit sphere in $\mathbf{C}^2$. Points of $\mathbf{C}^2$ will be written (z,w).

THEOREM. *Let $P(z,w) = w\bar{z}\bar{w}^2 + iz\bar{z}^2\bar{w}$ and put $F(z,w) = (z,w,P(z,w))$. Then F is a totally real embedding of S^3 in $\mathbf{C}^3$.*

PROOF. Instead of simply verifying that this P does what is needed we shall indicate the reasoning by which we found it.

The first part of the argument does not depend on the fact that we are dealing with S^3. Accordingly, let M be a smooth submanifold of $\mathbf{C}^2$,

$$M = \{(z,w) : \rho(z,w) = 0\},$$

where the gradient of the defining function ρ has no zero on M. Suppose that $F = (f_1, f_2, f_3)$ is a smooth embedding of M into $\mathbf{C}^3$. In order for F to be totally real it is necessary and sufficient that, at each point of $F(M)$, the tangent space to $F(M)$ contain three vectors that are linearly independent over $\mathbf{C}$.

At each point p of M consider the following tangent vectors to M:

$$t = \begin{pmatrix} i\partial\rho/\partial\bar{z} \\ i\partial\rho/\partial\bar{w} \end{pmatrix}, \quad u = \begin{pmatrix} i\partial\rho/\partial w \\ -i\partial\rho/\partial z \end{pmatrix}, \quad v = \begin{pmatrix} \partial\rho/\partial w \\ -\partial\rho/\partial z \end{pmatrix}.$$

They are mutually perpendicular (as vectors in $\mathbf{R}^4$), hence span the tangent space and give rise to corresponding directional derivatives D_t, D_u, D_v, where

$$(D_t f)(p) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[f(p+\varepsilon t) - f(p)],$$

and $D_u f, D_v f$ are defined similarly. Note that ε is *real* in these quotients.

Returning to $F = (f_1, f_2, f_3)$ we see that F is a totally real embedding if and only if the vectors $D_t F, D_u F, D_v F$ are linearly independent over $\mathbf{C}$ at all points of

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M, i.e., if and only if the determinant

$$\begin{vmatrix} D_t f_1 & D_u f_1 & D_v f_1 \\ D_t f_2 & D_u f_2 & D_v f_2 \\ D_t f_3 & D_u f_3 & D_v f_3 \end{vmatrix}$$

has no zero on M.

Let us now choose $f_1 = z$, $f_2 = w$, $f_3 = P$, where P has to be chosen. Certainly, F is then an embedding, and the determinant becomes

$$\begin{vmatrix} i\partial\rho/\partial\bar{z} & i\partial\rho/\partial w & \partial\rho/\partial w \\ i\partial\rho/\partial\bar{w} & -i\partial\rho/\partial z & -\partial\rho/\partial z \\ D_t P & D_u P & D_v P \end{vmatrix}.$$

If we multiply the second column by i and add to the third we see that our determinant is equal to

$$\{|\partial\rho/\partial z|^2 + |\partial\rho/\partial w|^2\} \cdot 2LP$$

where $LP = \frac{1}{2}(D_v P + iD_u P)$.

A simple calculation shows that

$$L = \frac{\partial\rho}{\partial\bar{w}}\frac{\partial}{\partial\bar{z}} - \frac{\partial\rho}{\partial\bar{z}}\frac{\partial}{\partial\bar{w}},$$

i.e., L is the tangential Cauchy-Riemann operator on M [**2**, p. 389]. We conclude that *the embedding $F(z,w) = (z,w,P(z,w))$ is totally real if and only if $LP \neq 0$ at every point of M.*

Now we restrict our attention to the case $M = S^3$, where $L = w\partial/\partial\bar{z} - z\partial/\partial\bar{w}$, *and we have to produce a P for which LP has no zero on S^3.*

How to do this is suggested by the action of L on the spaces $H(p,q)$; these consist of all harmonic homogeneous polynomials on $\mathbf{C}^2$ that have total degree p in (z,w) and total degree q in $(\bar{z},\bar{w})$: *L is a linear one-to-one map of $H(p,q)$ onto $H(p+1,q-1)$ if $q \geq 1$* [**2**, p. 398]. Similarly, the operator $\overline{L} = \bar{w}\partial/\partial z - \bar{z}\partial/\partial w$ is an isomorphism of $H(p+1,q-1)$ onto $H(p,q)$, and one computes that $L\overline{L}f = -p(q+1)f$ for every $f \in H(p,q)$. Proposition 12.4.1 of [**2**] is helpful in this computation.

Thus, given $g \in H(p,q)$ where $p \geq 1$, we now know how to find $h \in H(p-1,q+1)$ so that $Lh = g$. Put

$$g = |z|^4 + |w|^4 - 4|z|^2|w|^2 + i(|w|^2 - |z|^2).$$

Then $g \in H(1,1) + H(2,2)$, and g has no zero on S^3, for if $\operatorname{Im} g = 0$ then $|w|^2 = |z|^2 = 1/2$, hence $\operatorname{Re} g = -1/2$. Also, $g = LP_0$ for

$$P_0 = w\bar{z}\bar{w}^2 - z\bar{z}^2\bar{w} + i\bar{z}\bar{w}.$$

We conclude: *$(z,w) \to (z,w,P_0(z,w))$ is a totally real embedding of S^3 into $\mathbf{C}^3$.*

We can replace P_0 by a homogeneous polynomial. Simply replace the last term $\bar{z}\bar{w}$ by $\bar{z}\bar{w}(z\bar{z} + w\bar{w})$, and note that $z\bar{z} + w\bar{w} = 1$ on S^3. If we then multiply by $(1-i)/2$ we obtain P as in the statement of the theorem. One can easily check that

$$LP = |w|^2(|w|^2 - 2|z|^2) - i|z|^2(|z|^2 - 2|w|^2)$$

and that this has no zero on S^3.

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