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ON SURFACES IN $\mathbf{R}^4$

WALTER SEAMAN

ABSTRACT. We provide answers (Theorem C) to some questions concerning surfaces in $\mathbf{R}^4$ and maps into the quadric Q_2 raised by D. Hoffman and R. Osserman.

Let S be an oriented surface immersed in $\mathbf{R}^4$. The Gauss map of S is the map G of S into $G(2,4)$, the Grassmannian of oriented two-planes in $\mathbf{R}^4$, given by $G(p) = T_pS$. $G(2,4)$ can be identified with Q_2, the complex quadric in $\mathbf{C}P^3$, and in turn Q_2 is biholomorhic to $\mathbf{C}P^1 \times \mathbf{C}P^1$. If we give $\mathbf{C}P^3$ the Fubini-Study metric of constant holomorphic sectional curvature 2, then the induced metric on Q_2 is given by

$$2|dw_1|^2/\left(1+|w_1|^2\right)^2 + 2|dw_2|^2/\left(1+|w_2|^2\right)^2,$$

where (w_1, w_2) are coordinates on $\mathbf{C} \times \mathbf{C}$, viewed as local coordinates on $\mathbf{C}P^1 \times \mathbf{C}P^1$ [**1**]. The metric $2|dw|^2/(1+|w|^2)^2$ is the metric on $\mathbf{C}$ induced by the map of $\mathbf{C}$ onto $S^2(1/\sqrt{2}) \subset \mathbf{R}^3$ given by $w \mapsto \sigma^{-1}(\sqrt{2}\,w)$, where σ^{-1} is inverse stereographic projection (with the sphere sitting on the xy-plane). Thus, Q_2 is isometric to $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$. In particular, if z is a local conformal parameter on S, then *any* map G of S into Q_2 splits into a pair of maps $G(z) = (f_1(z), f_2(z))$, where $w_i = f_i(z)$ as above. Now define the following quantities on S for $i = 1, 2$:

$$F_i := \frac{f_{i\bar z}}{1+|f_i|^2}, \quad T_i(z) = \left[\frac{(f_i)_{z\bar z}}{(f_i)_{\bar z}} - \frac{2\bar f_i f_{iz}}{1+|f_i|^2}\right]_{\bar z} \quad \text{where } f_{i\bar z} \neq 0$$

with the usual z and $\bar z$ derivative notation. The following results are from [**1, 2**].

THEOREM A. *For the Gauss map G of an oriented surface S immersed in* $\mathbf{R}^4$, *write $G = (f_1(z), f_2(z))$ as above. Then we necessarily have*

$$|F_1| \equiv |F_2|, \tag{1}$$

and

$$\operatorname{Im}\{T_1 + T_2\} \equiv 0. \tag{2}$$

THEOREM B. *Let S_0 be a simply connected Riemann surface* (*here and subsequently*), *let $G = (f_1(z), f_2(z))$ be some map of S_0 into Q_2, and define F_i and T_i as before, where z is a conformal parameter on S_0.*

(i) *If $F_1 = F_2 \equiv 0$, then G is the Gauss map of a minimal surface in* $\mathbf{R}^4$, *provided S_0 is not compact.*

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(ii) *If F_1, F_2 are never zero, then G is the Gauss map of a surface S in $\mathbf{R}^4$ given by a conformal immersion of S_0 if and only if*

$$|F_1| \equiv |F_2|, \tag{1}$$

and

$$\operatorname{Im}\{T_1 + T_2\} \equiv 0. \tag{2}$$

Furthermore, in this case S is uniquely determined up to translation and homothety of $\mathbf{R}^4$.

Let (1′) denote the condition that F_1, F_2 are never zero (i.e., $f_{1\bar{z}}$ and $f_{2\bar{z}}$ are never zero) and $|F_1| \equiv |F_2|$. A special class of maps which satisfy (2) are harmonic maps, i.e., those $f(z)$'s such that

$$L(f) := f_{z\bar{z}} - 2\bar{f} f_z f_{\bar{z}} / \left(1 + |f|^2\right) = 0.$$

In particular, if

$$L(f_i) \equiv 0, \qquad i = 1, 2, \tag{3}$$

then (2) is automatically satisfied. Condition (3) is simply that the map $G\colon S_0 \to Q_2$ is harmonic. A theorem of Ruh and Vilms [**4**] asserts that the Gauss map of a submanifold of $\mathbf{R}^n$ is harmonic if and only if the submanifold has parallel mean curvature. Combining this with Theorem B we now have the following observation:

A map $G\colon S_0 \to Q_2$ is the Gauss map of a conformal immersion with parallel (nonzero) mean curvature in $\mathbf{R}^4$ if and only if (1′) and (3) hold.

Finally, an interesting subclass of surfaces of parallel mean curvature in $\mathbf{R}^4$ are minimal surfaces in some $S^3(r)$. Hoffman and Osserman also prove the following

PROPOSITION. *A map $G\colon S_0 \to Q_2$ is the Gauss map of a conformal minimal immersion of S_0 into some $S^3(r)$ (viewed as sitting in $\mathbf{R}^4$) if and only if* (1) *and* (3) *are satisfied, as well as the following*

$$f_{1z}/f_{1\bar{z}} = f_{2z}/f_{2\bar{z}}. \tag{4}$$

In view of these results, the following questions present themselves [**1**, **2**]: Given a map from S_0 into $S^2(1/\sqrt{2})$, represented locally by $f_1(z)$ as above, does there exist a map from S_0 into $S^2(1/\sqrt{2})$, represented by $f_2(z)$, such that the pair $(f_1(z), f_2(z))$ satisfies

Q1. (1′) *and* (2)? Suppose f_1 satisfies $L(f_1) = 0$. Does there exist $f_2(z)$ such that the pair (f_1, f_2) satisfies

Q2. (1′) *and* (3), *or*

Q3. (1′), (3) *and* (4)?

An affirmative answer to Q1 (Q2) would mean that the pair (f_1, f_2) is the Gauss map of a conformal immersion (with parallel nonzero mean curvature) of S_0 in $\mathbf{R}^4$, while an affirmative answer to Q3 would mean that the pair (f_1, f_2) is the Gauss map of a conformal minimal immersion of S_0 into some $S^3(r)$ (viewed as sitting in $\mathbf{R}^4$).

We answer Q2 and Q3 affirmatively in Theorem C. While this provides an affirmative answer to Q1 under the special assumption of (3) ($\Rightarrow$ (2)), we do not know the answer to Q1 in general.

THEOREM C. *Given a map from S_0, not conformally equivalent to S^2, into $S^2(1/\sqrt{2})$, written as $f_1(z)$ as above, such that $f_{1\bar{z}}$ is never zero, and $L(f_1) = 0$, there exists a one-parameter family of maps of S_0 into $S^2(1/\sqrt{2})$, written as $f_\theta(z)$, such that the pair (f_1, f_θ) satisfies* (1) *and* (3). *Furthermore, there is a unique θ_0 such that the pair (f_1, f_{θ_0}) also satisfies* (4). *If S_0 is conformally equivalent to S^2, $f_2 = f_1$ is the only possibility for even* (1) *and* (3).

REMARK. The idea of the proof is to regard f_1 as the Gauss map of a surface S of constant (nonzero) mean curvature in $\mathbf{R}^3$. The f_θ's are the Gauss maps of the associated family S_θ, $0 \leqslant \theta \leqslant 2\pi$, to S. It turns out that condition (4) is then satisfied exactly for the surface S_π:

PROOF OF THEOREM C. We regard $f_1(z)$ as the representation of a map of S_0 with $S^2(1)$ as follows: Let $\sigma(\sigma')$ be stereographic projection of $S^2(1/\sqrt{2})$ ($S^2(1)$) onto $\mathbf{C}$, and consider the transformation $\mathbf{C} \xrightarrow{\phi} \mathbf{C}$ by

$$\phi(w) = \tfrac{1}{2}\left(\sigma'\left(\sqrt{2}\left(\sigma^{-1}(\sqrt{2}\,w)\right)\right)\right).$$

ϕ is just the identity map on $\mathbf{C}$, so $\phi(f_1) = f_{1'}$. Replacing f_1 by $\tilde{f}_1 = \sqrt{2}(\sigma^{-1}(\sqrt{2}f_1)) \in S^2(1)$, and then representing $\tilde{f}_1$ by $\frac{1}{2}\sigma'(\tilde{f}_1)$, we see that we may regard f_1 as a map into $\mathbf{C}P^1$, with the metric $4|dw|^2/(1+|w|^2)^2$ of constant curvature 1. Thus it suffices to prove Theorem C with $S^2(1/\sqrt{2})$ replaced by $S^2(1)$. Now the conditions $f_{1\bar{z}} \neq 0$, $L(f_1) = 0$ mean that f_1 is a harmonic, nowhere anticonformal map of S_0 into $S^2 = S^2(1)$. From Hoffman-Osserman [1] and Kenmotsu [3], this guarantees that f_1 is the Gauss map of a conformal immersion X of S_0 into $\mathbf{R}^3$ with constant nonzero mean curvature. If we specify that $X(S_0)$ have constant mean curvature 1, then this determines $\mathscr{S}_0 = X(S_0)$ up to translation in $\mathbf{R}^3$. If S^0 is conformally equivalent to S^2, then $\mathscr{S}_0$ is the standard unit sphere, and any f_2: $S_0 \to S^2(1)$ satisfying (1) and (3) must come from the same X (up to translation of $\mathbf{R}^3$). For S_0 not conformally S^2, in the (global) isothermal parameter z, the metric induced on S_0 is $4F_0|dz|^2$ [3], where we have relabelled f_1 as f_0. Now consider the associate family $\mathscr{S}_\theta$ [5] to $\mathscr{S}_0$ $(0 \leqslant \theta < 2\pi)$. Then the Gauss maps f_θ of $\mathscr{S}_\theta$ satisfy (3), since each $\mathscr{S}_0$ has constant mean curvature. Since the metric on S_0 inherited from $\mathscr{S}_\theta$ is given by

$$4|f_{\theta\bar{z}}|^2/\left(1+|f_\theta|^2\right)^2|dz|^2,$$

and since $\mathscr{S}_\theta$ is isometric to $\mathscr{S}_0$, we also have condition (1) satisfied for the pair (f_0, f_θ). Let β^θ be the second fundamental form of $\mathscr{S}_\theta$. Then from formula 5.3 of [3], we have

$$\text{(5)} \qquad \frac{1}{F_\theta}\left\{\frac{\beta_{11}^\theta - \beta_{22}^\theta}{2} - i\beta_{12}^\theta\right\} = \frac{f_{\theta z}}{f_{\theta\bar{z}}}.$$

From this we see that condition (4), in the presence of (1), is equivalent to

$$\beta_{11}^{0} - \beta_{22}^{0} = \beta_{22}^{\theta} - \beta_{11}^{\theta}, \qquad \beta_{12}^{0} = -\beta_{12}^{\theta}. \tag{6}$$

Finally, since

$$\beta_{11}^{\theta} = \cos\theta\left(\beta_{11}^{0} - F_0\right) + \sin\theta\beta_{12}^{0} + F_0,$$
$$\beta_{22}^{\theta} = -\cos\theta\left(\beta_{11}^{0} - F_0\right) - \sin\theta\beta_{12}^{0} + F_0,$$
$$\beta_{12}^{\theta} = \cos\theta\beta_{12}^{0} - \sin\theta\left(\beta_{11}^{0} - F_0\right)$$

and $\beta_{11}^{0} + \beta_{22}^{0} = 2F_0$ [**5**], we see that condition (4) (cf. (6)) is equivalent to $\theta = \pi$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242