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ON THE EXISTENCE OF UNIFORMLY CONSISTENT ESTIMATES[1]

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ABSTRACT. Let $\mathscr{M}$ be a family of probability measures on $(\mathscr{X}, \mathscr{A})$ and U the uniform structure defined by vicinities of the form

$$\left\{(P, Q):\ \sup_{1 \leqslant i \leqslant K} |P^n(A_{i,n}) - Q^n(A_{i,n})| < \varepsilon\right\},$$

where P^n is the product measure on $(\mathscr{X}^n, \mathscr{A}^n)$, $A_{i,n} \in \mathscr{A}^n$, $\varepsilon > 0$, $n \wedge K \geqslant 1$. Let $\phi^*: (\mathscr{M}, U) \to (\phi^*(\mathscr{M}), d)$, where

$$d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_{L_1} = 2 \sup_{A \in \mathscr{A}} |P(A) - Q(A)|.$$

We consider the case where the space of measures M is L_1 separable and relate the existence of uniformly consistent estimates for $\phi^*(P)$ with uniform continuity of ϕ^* and L_1-total boundedness of M.

1. Introduction, notation and definitions. Let $(\mathscr{X}, \mathscr{A})$ be a space with a σ-field, let $\mathscr{M}$ be a family of probability measures on $\mathscr{A}$, $(\mathscr{X}^n, \mathscr{A}^n)$ the nth product space and σ-field, and let $X_1, \ldots, X_n$ be independent identically distributed observations according to some measure $P \in \mathscr{M}$, P^n being the nth product measure. Let Θ be a topological space which is homeomorphic to a subset of $[0,1]^\infty$, h being the homeomorphism, and let ρ be a metric on $[0,1]^\infty$ of the form

$$\rho(x, y) = \sum_{m=1}^{\infty} 2^{-m}|x_m - y_m|,$$

where x_m, y_m are coordinates of x, y respectively. Let ϕ^*: $P \to \phi^*(P)$ be a function defined on $\mathscr{M}$ with values in Θ, and $\phi = h \circ \phi^*$, $\phi: \mathscr{M} \to ([0,1]^\infty, \rho)$.

LeCam and Schwartz **[1960]** gave necessary and sufficient conditions for the existence of uniformly consistent estimates of $\phi(P)$ in terms of the uniform continuity of ϕ with respect to a uniform structure $U = \bigcup_n U_n$, where each U_n consists of vicinities of the form $\{(P, Q)$: $\sup_{1 \leqslant j \leqslant l} |\int f_j dP^n - \int f_j dQ^n| < 1$ for $f_1, \ldots, f_l$ bounded measurable functions on the product space $(\mathscr{X}^n, \mathscr{A}^n)\}$. The same uniform structure has been used by Pfanzagl **[1968]** and Moché **[1977]**. Under the above set-up we will explain why it is natural to consider the uniform structure U and we

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will offer a theorem on the existence of estimates. Relaxing the condition of homeomorphism of Θ with a subset of $[0,1]^{\infty}$ and assuming only that $(\phi^*(\mathscr{M}), d)$ is separable when metrized with the total variation (L_1) norm between the measures, i.e. $d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_{L_1} = 2\sup\{|P(A) - Q(A)|;\ A \in \mathscr{A}\}$, we offer a theorem of the same type in the form of equivalent propositions.

DEFINITION. We shall say ϕ is *uniformly consistently estimable* if there is a sequence T_n of measurable functions from $(\mathscr{X}^n, \mathscr{A}^n)$ to $([0,1]^{\infty}, \rho)$ such that for every $\varepsilon > 0$, $\sup\{P^n[\rho(T_n, \phi(P)) > \varepsilon];\ P \in \mathscr{M}\} \to 0$ as n tends to infinity.

REMARK. One can easily see that since ρ is bounded, the above definition is equivalent to $\sup\{E_{P^n}\rho(T_n, \phi(P)) \colon P \in \mathscr{M}\} \to 0$ as n tends to infinity.

Recall from topology the following

DEFINITION. Let (Y, d) be a metric space with a metric d. We say Y is *d-totally bounded* if for every $\varepsilon > 0$ there are $y_1, \ldots, y_{n(\varepsilon)}$ elements of Y such that

$$Y = \bigcup_{i=1}^{n(\varepsilon)} \{y \colon d(y, y_i) < \varepsilon\}.$$

In our effort to relax the hypothesis of homeomorphism of Θ with $[0,1]^{\infty}$ our estimator will take values in an abstract space. Terminology and results under this set-up concerning measurability (strong and weak), convergence (almost sure and almost uniform) and Bochner integration can be found in Hille and Phillips [**1957**, §§3.5 and 3.7]. For the notion of uniform structure the reader is referred to Choquet [**1969**, Chapter 2, §5]. For a deep insight on consistency questions, we suggest the remarkable paper of Kraft [**1955**].

2. A note on the LeCam-Schwartz set-up.

LEMMA 1. *Assume ϕ is uniformly consistently estimable by T_n. Then for every $\varepsilon > 0$ there exist positive integers $n = n(\varepsilon)$, $K = K(\varepsilon)$ and $l = l(\varepsilon)$, and measurable sets $B_1^n, \ldots, B_l^n$ of the nth product σ-field $\mathscr{A}^n$ such that*

$$\rho(\phi(P), \phi(Q)) \leqslant \varepsilon + K \cdot \sup\left\{\left|P^n(B_j^n) - Q^n(B_j^n)\right|;\ 1 \leqslant j \leqslant l(\varepsilon)\right\}.$$

PROOF. We will use the triangular inequality and then Jensen's inequality,

$$\begin{aligned}(1)\quad \rho(\phi(P), \phi(Q)) &\leqslant \rho(\phi(P), E_{P^n}T_n) + \rho(E_{P^n}T_n, E_{Q^n}T_n) + \rho(E_{Q^n}T_n, \phi(Q))\\ &\leqslant E_{P^n}\rho(\phi(P), T_n) + \rho(E_{P^n}T_n, E_{Q^n}T_n) + E_{Q^n}\rho(T_n, \phi(Q))\\ &\leqslant 2 \cdot \frac{\varepsilon}{4} + \rho(E_{P^n}T_n, E_{Q^n}T_n)\end{aligned}$$

for every $n \geqslant n(\varepsilon)$ by uniform convergence of T_n to $\phi(P)$. For the rest of the proof let $n = n(\varepsilon)$.

Consider

$$(2)\qquad \rho(E_{P^n}T_n, E_{Q^n}T_n) = \sum_{m=1}^{\infty} 2^{-m}\left|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m\right|,$$

where $(T_n)_m$ are the coordinates of T_n, $1 \leqslant m < \infty$. By assumption $|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m| \leqslant 2$, so there is $m_0 = m_0(\varepsilon) > 0$ such that

$$\sum_{m=m_0+1}^{\infty} 2^{-m}|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m| \leqslant \sum_{m=m_0+1}^{\infty} 2^{-m+1} < \frac{\varepsilon}{4}.$$

So now (2) becomes

$$\rho(E_{P^n}T_n, E_{Q^n}T_n) \leqslant \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m}|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m|. \tag{3}$$

Consider now $(T_n)_m$ for some $m \in \{1,\ldots,m_0\}$. Since $0 \leqslant (T_n)_m \leqslant 1$ for $\varepsilon/8$ there exists a simple function of the form $\sum_{i=1}^{l_m(\varepsilon)} a_{i,m} I_{A^n_{i,m}}$ such that

$$\sup\left\{\left|(T_n(x_1,\ldots,x_n))_m - \sum_{i=1}^{l_m(\varepsilon)} a_{i,m} I_{A^n_{i,m}}\right|; (x_1,\ldots,x_n) \in \mathscr{X}^n\right\} < \frac{\varepsilon}{8}.$$

The mth term of the sum in (3) becomes

$$|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m| \leqslant \frac{2\varepsilon}{8} + \sum_{i=1}^{l_m(\varepsilon)} a_{i,m}|P^n(A^n_{i,m}) - Q^n(A^n_{i,m})|$$

$$\leqslant \frac{\varepsilon}{4} + \left(\sum_{i=1}^{l_m(\varepsilon)} |a_{i,m}|\right) \cdot \sup\{|P^n(A^n_{i,m}) - Q^n(A^n_{i,m})|; 1 \leqslant i \leqslant l_m(\varepsilon)\}.$$

Letting $K_m(\varepsilon) = \sum_{i=1}^{l_m(\varepsilon)}|a_{i,m}|$ and repeating the same argument for all $m \in \{1,\ldots,m_0\}$ in (3) we get

$$\rho(E_{P^n}T_n, E_{Q^n}T_n) \leqslant \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m} \cdot \frac{\varepsilon}{4}$$

$$+ \sum_{m=1}^{m_0} 2^{-m} \cdot K_m(\varepsilon) \cdot \sup\{|P^n(A^n_{i,m}) - Q^n(A^n_{i,m})|; 1 \leqslant i \leqslant l_m(\varepsilon)\}$$

$$\leqslant \frac{\varepsilon}{2} + \left(\sum_{m=1}^{m_0} 2^{-m}K_m(\varepsilon)\right) \cdot \sup\Big\{|P^n(B^n_j) - Q^n(B^n_j)|; 1 \leqslant j \leqslant \sum_{m=1}^{m_0} l_m(\varepsilon);$$

$$B^n_j \in \{A^n_{i,m}, 1 \leqslant m \leqslant m_0, 1 \leqslant i \leqslant l_m(\varepsilon)\}\Big\}.$$

By letting $K(\varepsilon) = \sum_{m=1}^{m_0} 2^{-m}K_m(\varepsilon)$, $l(\varepsilon) = \sum_{m=1}^{m_0} l_m(\varepsilon)$ and replacing in (1) we get

$$\rho(\phi(P), \phi(Q)) \leqslant \varepsilon + K \cdot \sup\{|P^n(B^n_j) - Q^n(B^n_j)|; 1 \leqslant j \leqslant l\}.$$

Q.E.D.

The above lemma shows that giving $\mathscr{M}$ the uniform structure consisting of vicinities of the form $\{(P,Q)\colon \sup\{|P^n(A^n_i) - Q^n(A^n_i)|; 1 \leqslant i \leqslant l\} < \varepsilon\}$ for $l \in N^+$, $n \in N^+$, $\varepsilon > 0$, is the natural way for associating uniform convergence in probability of T_n with the uniform continuity of $\phi\colon \mathscr{M} \to ([0,1]^\infty, \rho)$.

It is now easy to see that these vicinities give rise to the same uniform structure U as those in the introduction.

We offer the following

LEMMA 2. *Let* ϕ: $(\mathscr{M}, U) \to ([0,1]^\infty, \rho)$. *The following are equivalent*:

(1) ϕ *is uniformly continuous.*

(2) *For every* $\varepsilon > 0$ *there exists* $K(\varepsilon)$, $m(\varepsilon)$, $n(\varepsilon)$ *all positive, and* $B_1^{m(\varepsilon)}, \dots, B_{n(\varepsilon)}^{m(\varepsilon)}$ *elements of the* $m(\varepsilon)$*-product* σ*-field* $\mathscr{A}^{m(\varepsilon)}$ *such that*

$$\rho(\phi(P), \phi(Q)) \leqslant \varepsilon + K(\varepsilon) \cdot \sup\left\{\left|P^{m(\varepsilon)}\left(B_i^{m(\varepsilon)}\right) - Q^{m(\varepsilon)}\left(B_i^{m(\varepsilon)}\right)\right| : 1 \leqslant i \leqslant n(\varepsilon)\right\}.$$

PROOF. (2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (2). Since ϕ is uniformly continuous, for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, $m = m(\varepsilon) > 0$, $n = n(\varepsilon) > 0$ and sets $B_1^m, \dots, B_n^m$ in $\mathscr{A}^m$ such that if

$$\sup\left\{\left|P^m(B_i^m) - Q^m(B_i^m)\right|; 1 \leqslant i \leqslant n\right\} < \delta(\varepsilon),$$

then $\rho(\phi(P), \phi(Q)) < \varepsilon$.

From this, for every P, Q such that $\rho(\phi(P), \phi(Q)) \geqslant \varepsilon$ it is necessary that $\sup\{|P^m(B_i^m) - Q^m(B_i^m)|; 1 \leqslant i \leqslant n\} \geqslant \delta(\varepsilon)$.

Going to the statement it is required to prove: Note that for every $\varepsilon > 0$ either $\rho(\phi(P), \phi(Q)) < \varepsilon$ or $\rho(\phi(P), \phi(Q)) \geqslant \varepsilon$. For the second case we have then

$$\begin{aligned}\rho(\phi(P), \phi(Q)) \leqslant \rho(\phi(P), \phi(Q)) \cdot \Big[\inf\Big\{\sup\left\{\left|P^m(B_i^m) - Q^m(B_i^m)\right|; 1 \leqslant i \leqslant n\right\};\\ P, Q\colon \rho(\phi(P), \phi(Q)) > \varepsilon\Big\}\Big]^{-1}\\ \cdot \sup\left\{\left|P^m(B_i^m) - Q^m(B_i^m)\right|; 1 \leqslant i \leqslant n\right\}\\ \leqslant 2[\delta(\varepsilon)]^{-1} \sup\left\{\left|P^m(B_i^m) - Q^m(B_i^m)\right|; 1 \leqslant i \leqslant n\right\}.\end{aligned}$$

Let $2[\delta(\varepsilon)]^{-1} = K(\varepsilon)$. So finally for all P, Q

$$\rho(\phi(P), \phi(Q)) \leqslant \varepsilon + K(\varepsilon) \cdot \sup\left\{\left|P^m(B_i^m) - Q^m(B_i^m)\right|; 1 \leqslant i \leqslant n\right\}.$$

We offer

THEOREM 1. *Let* ϕ: $(\mathscr{M}, U) \to ([0,1]^\infty, \rho)$ *with* U *defined as above. The following statements are equivalent*:

(1) ϕ *is uniformly continuous.*

(2) *There exists a uniformly consistent estimator of* $\phi(P)$.

PROOF. (1) $\Rightarrow$ (2). Given in LeCam and Schwartz [**1960**, Theorem 1].

(2) $\Rightarrow$ (1). Corollary of Lemmas 1 and 2.

3. The main result. Consider now the case ϕ^*: $(\mathscr{M}, U) \to (\phi^*(\mathscr{M}), \|\cdot\|_{L_1})$, as in the introduction, with $\phi^*(\mathscr{M})$ separable when metrized with the L_1-distance between measures. We will give our result in a series of lemmas.

LEMMA 3. *The uniform structure* U *is precompact* (*in the sense that for any vicinity of the form* $\{(P, Q)\colon \sup\{|P^m(A_i^m) - Q^m(A_i^m)|; 1 \leqslant i \leqslant n\} < \varepsilon\}$ *there exist* $P_1, \dots, P_l$ *in* $\mathscr{M}$ *such that* $\mathscr{M} = \bigcup_{k=1}^{l}\{P\colon \sup\{|P^m(A_i^m) - P_k^m(A_i^m)|; 1 \leqslant i \leqslant n\} < \varepsilon\}$).

PROOF. Given in LeCam and Schwartz [**1960**, p. 142].

LEMMA 4. *Let* ϕ^*: $(\mathscr{M}, U) \to (\phi^*(\mathscr{M}), \|\cdot\|_{L_1})$. *The following propositions are equivalent*:

(A) ϕ^* *is uniformly continuous.*

(B) $\mathscr{M}$ *is* L_1*-totally bounded.*

PROOF. (B) $\Rightarrow$ (A). To prove ϕ^* is uniformly continuous, it suffices to prove that for every $\varepsilon > 0$ there exist $\delta(\varepsilon)$ and $B_1, \dots, B_n$ sets in $\mathscr{A}^m$ such that for every $(P, Q) \in V_{\delta(\varepsilon), m, B_1, \dots, B_n} = \{(\tilde{P}, \tilde{Q})\colon \sup\{|\tilde{P}^m(B_i) - \tilde{Q}^m(B_i)|;\ 1 \leqslant i \leqslant n\} < \delta(\varepsilon)\}$ implies $\|\phi^*(P) - \phi^*(Q)\| = \|P - Q\|_{L_1} < \varepsilon$.

Consider $\varepsilon/5 > 0$. By L_1-total boundedness of the space $\mathscr{M}$ there exist $P_1, \dots, P_k$ in $\mathscr{M}$ such that for every $P \in \mathscr{M}$, $\inf\{\|P - P_i\|_{L_1};\ 1 \leqslant i \leqslant k\} < \varepsilon$. On the other hand

$$\|P_i - P_j\|_{L_1} = P_i\left\{x\colon \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x)\right\} - P_j\left\{x\colon \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x)\right\},$$

where the existence of the dominating measure μ is secured by total boundedness. Consider all sets of the form $\{\{x\colon dP_i(x)/d\mu > dP_j(x)/d\mu\}$ for $1 \leqslant i < j \leqslant k\}$ and call them $B_1, \dots, B_n$.

Consider now $V_{\varepsilon/5, 1, B_1, \dots, B_n} = \{(\tilde{P}, \tilde{Q})\colon \sup\{|\tilde{P}(B_i) - \tilde{Q}(B_i)|;\ 1 \leqslant i \leqslant n\} < \varepsilon/5\}$. Let $P, Q \in V_{\varepsilon/5, 1, B_1, \dots, B_n}$. We will prove $\|P - Q\|_{L_1} < \varepsilon$. Using the triangular inequality we get

$$\begin{aligned}\|P - Q\|_{L_1} &\leqslant \|P - P_i\|_{L_1} + \|P_i - P_j\|_{L_1} + \|P_j - Q\| < \frac{2\varepsilon}{5} + \|P_i - P_j\|_{L_1}\\ &\leqslant \frac{4\varepsilon}{5} + \sup\left\{|P(B_i) - Q(B_i)|;\ 1 \leqslant i \leqslant n\right\} < \varepsilon.\end{aligned}$$

(A) $\Rightarrow$ (B). To prove now that $\mathscr{M}$ is L_1-totally bounded, it suffices to prove that for every $\varepsilon > 0$ there exist $P_1, \dots, P_l$ in $\mathscr{M}$ such that $\inf\{\|P - P_i\|;\ 1 \leqslant i \leqslant l\} < \varepsilon$ for every P in $\mathscr{M}$.

Fix $\varepsilon > 0$. Since ϕ^* is uniformly continuous for that ε there exist $\delta(\varepsilon)$, $m(\varepsilon)$ and $B_1, \dots, B_n$, in $\mathscr{A}^{m(\varepsilon)}$ such that if $(P, Q) \in V_{\delta(\varepsilon), m(\varepsilon), B_1, \dots, B_n}$, then $\|\phi^*(P) - \phi^*(Q)\| = \|P - Q\|_{L_1} < \varepsilon$. By Lemma 3, U is precompact so for $V_{\delta(\varepsilon), m(\varepsilon), B_1, \dots, B_n}$ there exist $P_1, \dots, P_l$ such that $\mathscr{M} = \bigcup_{k=1}^{l}\{P\colon \sup\{|P^{m(\varepsilon)}(B_i) - P_k^{m(\varepsilon)}(B_i)|;\ 1 \leqslant i \leqslant n\} < \delta(\varepsilon)\}$. From this it follows that $\inf\{\|P - P_i\|_{L_1};\ 1 \leqslant i \leqslant l\} < \varepsilon$ for every $P \in \mathscr{M}$.

LEMMA 5. *Assume now that* $\mathscr{M}$ *is* L_1*-separable*, ϕ^*: $(\mathscr{M}, U) \to (\phi^*(\mathscr{M}), \|\cdot\|_{L_1})$, *and* μ *is a dominating measure existing by* L_1*-separability assumption. The following propositions are equivalent*:

(B) $\mathscr{M}$ *is* L_1*-totally bounded.*

(C) (a) *There exists a uniformly consistent estimator for* $\phi^*(P)$ *with values in* $\phi^*(\mathscr{M})$, (b) *for every* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that if* $\mu(A) < \delta(\varepsilon)$, *then* $\sup\{P(A);\ P \in \mathscr{M}\} < \varepsilon$.

PROOF. (B) $\Rightarrow$ (C). We will first prove (b). Fix $\varepsilon > 0$. For $\varepsilon/2$ there exist $P_1, \dots, P_n$ in $\mathscr{M}$ such that $\inf\{\|P - P_i\|_{L_1};\ 1 \leqslant i \leqslant n\} < \varepsilon/2$ for every P in $\mathscr{M}$. By absolute continuity of $P_1, \dots, P_n$ with respect to μ there is a $\delta(\varepsilon) > 0$ such that if $\mu(A) < \delta(\varepsilon)$, then $\sup\{P_i(A);\ 1 \leqslant i \leqslant n\} < \varepsilon/2$. Consider now $P \in \mathscr{M}$. Then

$$P(A) = P(A) - P_i(A) + P_i(A) \leqslant \|P - P_i\|_{L_1} + P_i(A) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

where P_i is the center of the L_1-ball of radius $\varepsilon/2$ containing P and this holds for every $A \in \mathscr{A}$ such that $\mu(A) < \delta(\varepsilon)$ and for every P in $\mathscr{M}$.

We now prove (a) by constructing a uniformly consistent minimum distance estimator.

Let a_n be a sequence of numbers such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant \cdots \geqslant 0$ tending to 0. Since $\mathscr{M}$ is L_1-totally bounded for every a_n, there is an a_n-dense subset of measures $P_1, \ldots, P_{N(a_n)}$ in $\mathscr{M}$. Let

$$\mathscr{F}_{a_n} = \left\{ \left\{ x\colon \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \right\}, 1 \leqslant i < j \leqslant N(a_n) \right\}.$$

By applying the triangular inequality, we have then that for every P, Q in $\mathscr{M}$, $\|P - Q\|_{L_1} \leqslant 4a_n + 2\sup\{|P(A) - Q(A)|; A \in \mathscr{F}_{a_n}\}$.

Let $X_1, \ldots, X_k$ be independent identically distributed observations from $P \in \mathscr{M}$ and $\mu_k(A) = \frac{1}{k}\sum_{i=1}^k I_A(X_i)$ the empirical measure indexed by $A \in \mathscr{A}$. Define an estimator

$$\begin{aligned} \hat{P}_{k,n}\colon \sup\left\{ |\mu_k(A) - \hat{P}_{k,n}(A)|; A \in \mathscr{F}_{a_n} \right\} \\ = \inf\left\{ \sup\left\{ |\mu_k(A) - Q(A)|; A \in \mathscr{F}_{a_n} \right\}; Q \in \mathscr{M} \right\} \end{aligned}$$

(without loss of generality we can assume the infimum is achieved). For fixed a_n, by the law of large numbers, $\sup\{|\mu_k(A) - \hat{P}_{k,n}(A)|; A \in \mathscr{F}_{a_n}\} \leqslant \sup\{|\mu_k(A) - P(A)|; A \in \mathscr{F}_{a_n}\} \to 0$ as k tends to infinity. So there is a $K(a_n)$ such that

$$\sup\left\{ |\mu_k(A) - \hat{P}_{k,n}(A)|; A \in \mathscr{F}_{a_n} \right\} \leqslant a_n \quad \text{for } K \geqslant K(a_n).$$

Repeating the same construction for $\mathscr{F}_{a_{n+1}}$, there exists $K(a_{n+1})$ such that $\sup\{|\mu_k(A) - \hat{P}_{k,n+1}(A)|; A \in \mathscr{F}_{a_{n+1}}\} < a_{n+1}$ for $K \geqslant K(a_{n+1})$. Let $K(a_n, a_{n+1}) = \max\{K(a_n), K(a_{n+1})\}$ and define

$$\hat{P}_k = \begin{cases} \hat{P}_{k,n} & \text{for } K \leqslant K(a_n, a_{n+1}), \\ \hat{P}_{k,n+1} & \text{for } K(a_n, a_{n+1}) < K \leqslant K(a_{n+1}, a_{n+2}). \end{cases}$$

We claim that $\|\hat{P}_k - P\|_{L_1} \to 0$ as k tends to infinity in P^k-probability. By construction,

$$\begin{aligned} \|\hat{P}_k - P\|_{L_1} &\leqslant 4a_n + 2\sup\left\{ |\hat{P}_k(A) - P(A)|; A \in \mathscr{F}_{a_n} \right\} \\ &\leqslant 4a_n + 2\sup\left\{ |\mu_k(A) - \hat{P}_k(A)|; A \in \mathscr{F}_{a_n} \right\} \leqslant 5a_n \end{aligned}$$

for $K \geqslant K(a_n, a_{n+1})$ with high probability. Q.E.D.

(C) $\Rightarrow$ (B). By Lemma 4 it is enough to prove (C) $\Rightarrow$ (A), i.e. we will prove that ϕ^* is uniformly continuous.

By assumption there is a uniformly consistent estimator T_n of $\phi^*(P)$ taking values in $\phi^*(\mathscr{M})$. So for every n, there is $\hat{P}_n$ in $\mathscr{M}$ such that $T_n = \phi^*(\hat{P}_n)$. By uniform consistency we will then have

$$(4) \qquad \sup\left\{ E_{P^n}\|T_n - \phi^*(P)\|; P \in \mathscr{M} \right\} = \sup\left\{ E_{P^n}\|\hat{P}_n - P\|_{L_1}; P \in \mathscr{M} \right\} \to 0$$

as n tends to infinity.

Fix $\varepsilon > 0$ and consider

$$\begin{aligned}\|\phi^*(P) - \phi^*(Q)\| &= \|P - Q\|_{L_1}\\ &\leqslant \|P - E_{P^n}\hat{P}_n\|_{L_1} + \|E_{P^n}\hat{P}_n - E_{Q^n}\hat{P}_n\|_{L_1} + \|E_{Q^n}\hat{P}_n - Q\|_{L_1}\\ &\leqslant E_{P^n}\|P - \hat{P}_n\|_{L_1} + \|E_{P^n}\hat{P}_n - E_{Q^n}\hat{P}_n\|_{L_1} + E_{Q^n}\|\hat{P}_n - Q\|_{L_1}\\ &\leqslant 2\varepsilon + \|E_{P^n}\hat{P}_n - E_{Q^n}\hat{P}_n\|_{L_1}\end{aligned}$$

for all $n \geqslant n(\varepsilon)$ by (4). Let $n = n(\varepsilon)$ for the rest of the proof.

Observe now that $\hat{P}_n$ takes values in the space $\mathscr{M}$ which is L_1-separable. By separability there exists a dominating measure μ. Also strong measurability is equivalent to weak measurability (so we do not have any measurability problems) and there exist sets A_i^n in $\mathscr{A}^n$, $i = 1, 2, \ldots$, such that $\|\hat{P}_n - \sum_{i=1}^{\infty} I_{A_i^n} P_i\|_{L_1} < \varepsilon$ a.e. μ^n, with $\{P_i\}$, $1 \leqslant i \leqslant n$, being the countable dense subset of $\mathscr{M}$.

So we finally have

$$\|\phi^*(P) - \phi^*(Q)\| \leqslant 4\varepsilon + \left\| E_{P^n} \sum_{i=1}^{\infty} I_{A_i^n} P_i - E_{Q^n} \sum_{i=1}^{\infty} I_{A_i^n} P_i \right\|. \tag{5}$$

As we know, $\sum_{i=1}^{\infty} I_{A_i^n} P_i = \lim_{k\to\infty} \sum_{i=1}^{k} I_{A_i^n} P_i$ in L_1, μ^n a.s. and the convergence is almost uniform μ^n, so for $\delta(\varepsilon) > 0$ there exists E_ε in $\mathscr{A}^n$ such that $\mu^n(E_\varepsilon) < \delta(\varepsilon)$ and for $K = K(\varepsilon)$ big enough $\|\sum_{i=1}^{\infty} I_{A_i^n} P_i - \sum_{i=1}^{k} I_{A_i^n} P_i\|_{L_1} < \varepsilon$ for all x in $X^n - E_\varepsilon$.

We will try now to evaluate the right-hand side of (5):

$$\begin{aligned}&\left\| E_{P^n} \sum_{i=1}^{\infty} I_{A_i^n} P_i - E_{P^n} \sum_{i=1}^{k} I_{A_i^n} P_i \right\|\\ &\quad = \left\| \int_{E_\varepsilon} \left(\sum_{i=1}^{\infty} I_{A_i^n} P_i - \sum_{i=1}^{k} I_{A_i^n} P_i \right) dP^n + \int_{\mathscr{X}^n - E_\varepsilon} \left(\sum_{i=1}^{\infty} I_{A_i^n} P_i - \sum_{i=1}^{k} I_{A_i^n} P_i \right) dP^n \right\|_{L_1}\\ &\quad \leqslant 2P^n(E_\varepsilon) + \varepsilon \cdot P^n(\mathscr{X}^n - E_\varepsilon) \leqslant 3\varepsilon.\end{aligned}$$

Repeating the same type of calculation for $E_{Q^n} \sum_{i=1}^{\infty} I_{A_i^n} P_i$ and replacing in (5), we get

$$\begin{aligned}\|\phi^*(P) - \phi^*(Q)\| &\leqslant 10\varepsilon + \left\| E_{P^n} \sum_{i=1}^{k} I_{A_i^n} P_i - E_{Q^n} \sum_{i=1}^{k} I_{A_i^n} P_i \right\|_{L_1}\\ &= 10\varepsilon + \sup\left\{ \left| \sum_{i=1}^{k} \left(P^n(A_i^n) - Q^n(A_i^n)\right) \cdot P_i(A) \right|; A \in \mathscr{A} \right\}\\ &\leqslant 10\varepsilon + K \cdot \sup\left\{ |P^n(A_i^n) - Q^n(A_i^n)|; 1 \leqslant i \leqslant k \right\}. \quad \text{Q.E.D.}\end{aligned}$$

From Lemmas 4 and 5 we have the following

THEOREM 2. *Let $\mathscr{M}$ be an L_1-separable family of measures, ϕ^*: $(\mathscr{M}, U) \to (\phi^*(\mathscr{M}), \|\cdot\|_{L_1})$ and μ a dominating measure. The following positions are equivalent.*

(A) *ϕ^* is uniformly continuous.*

(B) *$\mathscr{M}$ is L_1-totally bounded.*

(C) (a) *There exists a uniformly consistent estimator for* $\phi^*(P)$ *with values in* $\phi^*(\mathscr{M})$, (b) *for every* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that for every* A *in* $\mathscr{A}$: $\mu(A) < \delta(\varepsilon)$, $\sup\{P(A); P \in \mathscr{M}\} < \varepsilon$.

REMARK. This theorem shows that in the case that $\mathscr{M}$ is uniformly dominated every uniformly consistent estimator in L_1 can be achieved through minimum distance. On the other hand the condition of uniform domination is not a necessary condition for the existence of uniformly consistent estimates in L_1 as the example of families of densities satisfying the Hoeffding-Wolfowitz [**1958**] condition shows.

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