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A REGULAR COUNTEREXAMPLE TO THE γ-SPACE CONJECTURE

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ABSTRACT. This paper presents a completely regular counterexample to the conjecture that every γ-space is quasi-metrizable. Junnila has shown that developable γ-spaces are quasi-metrizable; this example shows that "developable" cannot be replaced by "quasi-developable". In the process we provide a method for constructing non-n-pretransitive spaces.

1. Introduction. The γ-space conjecture is the conjecture that every γ-space is quasi-metrizable. This conjecture has been proven for various classes of spaces, among them spaces with orthobases, developable spaces, and suborderable spaces [**G-K2**; **J2**; **B-K1**]. In [**F**] it is shown how to construct a counterexample $\tilde{X}$ to the γ-space conjecture from a γ-space X having a neighbournet U with the property that U^k is not a normal neighbournet for any $k \in \mathbf{N}$ (such a space X may be obtained by taking the topological sum of a sequence of γ-spaces X_n, where X_n is not n-pretransitive). However, even if X is completely regular, $\tilde{X}$ need not be regular at all. A sufficient condition for $\tilde{X}$ to be completely regular is that X be completely regular and that U^k be a clopen neighbournet for every $k \in \mathbf{N}$.

We present here a construction which begins with a quasi-metrizable space Y of cardinality at most $\mathbf{c}$ which is not n^+-pretransitive, and yields another quasi-metrizable space $\hat{Y}$ of cardinality $\mathbf{c}$ that is not $(n+1)^+$-pretransitive. If U is a neighbournet on Y such that U^{n+} is not normal, then this construction yields a neighbournet $\hat{U}$ on $\hat{Y}$ such that $\hat{U}^{(n+1)+}$ is not normal (see Lemma 1 below). This construction will allow us to inductively generate a sequence of spaces $X_n = \hat{X}_{n-1}$ with neighbournets $U_n = \hat{U}_{n-1}$ such that U_n^{n+} is not normal. If we take X to be the topological sum $X = \bigcup_{n=0}^{\infty} X_n$ and $U = \bigcup_{n=0}^{\infty} U_n$, we may then apply [**F**] to construct a γ-space $\tilde{X}$ which is not quasi-metrizable.

By starting from a suitable space X_0, we can inductively guarantee that X_n will be completely regular and that U_n^k will be clopen for every $k \in \mathbf{N}$ (see Lemma 3 below). In this case, the counterexample $\tilde{X}$ will be completely regular, as intended.

2. Terminology. A *quasi-metric* is a generalized metric d satisfying the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ but not necessarily the symmetry axiom $d(x, y) = d(y, x)$ [**N**; **W**]. A space X is said to be *quasi-metrizable* if it has a

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compatible quasi-metric d—i.e. at each point $x \in X$ the sets $B_d(x; \varepsilon) = \{y: d(x, y) < \varepsilon\}$, for $\varepsilon > 0$, form a neighbourhood base.

We will use Junnila's neighbournet notation [**J1**]. A *neighbournet* on a space X is a binary relation V such that $V[x]$ is a neighbourhood of x for every $x \in X$. A neighbournet V is called *open*, *closed* or *clopen* if every $V[x]$ is open, closed or clopen, respectively. A sequence $\langle V_n: n \in \mathbf{N}\rangle$ of neighbournets is called *basic* if at each point $x \in X$ the sets $V_n[x]$, for $n \in \mathbf{N}$, form a neighbourhood base; and *normal* if $V_{n+1}^2 \subseteq V_n$ for each n. A neighbournet is said to be *normal* if it is a member of a normal sequence of neighbournets.

With this terminology, a T_1 space is quasi-metrizable if and only if it has a normal basic sequence of neighbournets [**R**; **J1**]. Similarly, a T_1 space is a γ-space if and only it has a sequence $\langle V_n: n \in \mathbf{N}\rangle$ of neighbournets such that the sequence $\langle V_n^2: n \in \mathbf{N}\rangle$ is basic [**H**; **LF**; **J1**]. Clearly every quasi-metrizable space is a γ-space.

If U is a binary relation on a space X we define a new relation U^+ on X by $U^+[x] = \bigcap\{U[G]: G \text{ is a neighbourhood of } x\}$. If U is a neighbournet then $U^n \subseteq (U^n)^+ \subseteq U^{n+1}$ for each nonnegative integer n [**K1**]. We will write U^{n+} for $(U^n)^+$.

A space X is called *n-pretransitive* (*n^+-pretransitive*) if whenever U is a neighbournet on X then U^n (U^{n+}) is a normal neighbournet [**FL**, p. 191, §6.21; cf. also **K1**]. The n^+-pretransitivity property lies strictly between n-pretransitivity and $(n+1)$-pretransitivity. Since $U^0[x] = \{x\}$, observe that a space is 0-pretransitive (0^+-pretransitive) if and only if it is discrete (the arbitrary intersection of open sets is open).

The importance of n- and n^+-pretransitivity is that an n- or n^+-pretransitive γ-space is quasi-metrizable [**FL**, p. 165, §7.19], and that almost all partial solutions to the γ-space conjecture have implicitly used this property: [**G**; **J2**; **K1**; **K2**] have all shown that the spaces concerned were 2- or 2^+-pretransitive.

3. The construction of $\hat{Y}$ and $\hat{U}$. Let Y be a quasi-metrizable space and $\langle V_n: n \in \mathbf{N}\rangle$ a normal basic sequence for Y. The structure of $\hat{Y}$ is as follows.

The points of $\hat{Y}$ are the points of $Y \times \mathbf{R}$. We presume that $\mathbf{R}$ is partitioned into sets A and B.

For each $b \in B$ we declare $Y \times \{b\}$ to be a clopen subspace of $\hat{Y}$ canonically homeomorphic to Y. If $\langle y, b\rangle \in Y \times \{b\}$ we define $\hat{V}_n[\langle y, b\rangle] = V_n[y] \times \{b\}$.

We presume that Z is a chosen subset of $Y \times A$; and for each $\langle x, a\rangle \in Z$ that $S(x, a)$ is a chosen subset of $Y \times B$. We define the basic neighbourhoods of $\langle x, a\rangle \in Z$ to be

$$\hat{V}_n[\langle x, a\rangle] = \{\langle x, a\rangle\} \cup \bigcup\{V_n[y] \times \{b\}: \langle y, b\rangle \in S(x, a) \text{ and } |b - a| < 2^{-n}\}.$$

All points $\langle x, a\rangle$ in $X \times A$ which are not in Z are isolated; for these points we define $\hat{V}_n[\langle x, a\rangle] = \{\langle x, a\rangle\}$.

The structure of $\hat{Y}$ as outlined above does not by itself guarantee that $\hat{Y}$ will be Hausdorff, even if Y is. However, $\hat{Y}$ will be T_1, and so the Hausdorff property will be guaranteed if $\hat{Y}$ is regular. It is not difficult to show that $\langle \hat{V}_n: n \in \mathbf{N}\rangle$ is a normal basic sequence for $\hat{Y}$; and hence $\hat{Y}$ is quasi-metrizable.

The structure of $\hat{U}$ is as follows. We presume that A is further partitioned into sets A_p ($p \in \mathbf{N}$), and that Z_p denotes the set of all points $\langle x, a\rangle \in Z$ with $a \in A_p$. Define $\hat{U}$ by

$$\begin{aligned} \hat{U}[\langle x, b\rangle] &= U[x] \times \{b\} && \text{if } b \in B; \\ \hat{U}[\langle x, a\rangle] &= \hat{V}_p[\langle x, a\rangle] && \text{if } \langle x, a\rangle \in Z_p, p \in \mathbf{N}; \\ \hat{U}[\langle x, a\rangle] &= \{\langle x, a\rangle\} && \text{otherwise.} \end{aligned}$$

Observe that if $\langle x, a\rangle \in Z$, then

$$\begin{aligned} \hat{U}^{n+}[\langle x, a\rangle] &= \hat{U}^n[\langle x, a\rangle] \cup \bigcap_{k=1}^{\infty} \bigcup \{U^n \circ V_k[y] \times \{b\} : \langle y, b\rangle \in S(x, a) \\ &\qquad\qquad \text{and } |b - a| < 2^{-k}\} \\ &= \hat{U}^n[\langle x, a\rangle]. \end{aligned}$$

Note that the construction of $\hat{Y}$ and $\hat{U}$ depends on the choices made of $A = \bigcup_{p=1}^{\infty} A_p$, B, Z, and $S(x, a)$ for each $\langle x, a\rangle \in Z$. We will elaborate later on how these choices are to be made. The lemmas below discuss the properties required of $\hat{Y}$ and $\hat{U}$.

LEMMA 1. *Suppose that U^{n+} is not a normal neighbournet on Y. If $\hat{Y}$ is constructed so that*

(I) *B is a dense Baire subset of* $\mathbf{R}$;

(II) *if E is a subset of $Y \times B$, and the canonical projection of E onto B is somewhere dense in* $\mathbf{R}$, *then for every $p \in \mathbf{N}$ there is a point $\langle x, a\rangle \in Z_p$ such that $\langle x, a\rangle \in$* $\mathrm{cl}(E \cap S(x, a))$; *and*

(III) *for each $\langle x, a\rangle \in Z$ and each $b \in B$ there is at most one point of $S(x, a)$ in $Y \times \{b\}$;*

then $\hat{U}^{(n+1)+}$ will not be a normal neighbournet on $\hat{Y}$.

PROOF. Let W be a normal neighbournet on $\hat{Y}$. To show that $\hat{U}^{(n+1)+}$ is not normal, we will show that $W^2 \not\subseteq \hat{U}^{(n+1)+}$.

Because U^{n+} is not normal, we may find for each $b \in B$ a point $y_b \in Y$ such that $W[\langle y_b, b\rangle] \not\subseteq U^{n+}[y_b] \times \{b\}$. Let G_b be a neighbourhood of y_b in Y such that $W[\langle y_b, b\rangle] \not\subseteq U^n[G_b] \times \{b\}$.

By (I), we may find a fixed $p \in \mathbf{N}$ and a subset D of B which is somewhere dense in $\mathbf{R}$ such that $V_p[y_b] \subseteq G_b$ for all $b \in D$.

By (II), there exists a point $\langle x, a\rangle \in Z_p$ such that $\langle x, a\rangle \in \mathrm{cl}(\{\langle y_b, b\rangle : b \in D\} \cap S(x, a))$. Note that

$$\begin{aligned} \hat{U}^{(n+1)+}[\langle x, a\rangle] &= \hat{U}^{n+1}[\langle x, a\rangle] = \hat{U}^n \circ \hat{V}_p[\langle x, a\rangle] \\ &= \{\langle x, a\rangle\} \cup \bigcup \left\{U^n \circ V_p[y] \times \{b\} : \langle y, b\rangle \in S(x, a) \text{ and } |b - a| < 2^{-p}\right\}. \end{aligned}$$

Choose some $\langle y_b, b\rangle \in W[\langle x, a\rangle] \cap S(x, a)$ so that $b \in D$. Then $U^n \circ V_p[y_b] \subseteq U^n[G_b]$ and hence we may find some point $\langle z, b\rangle \in W[\langle y_b, b\rangle] \setminus U^n \circ V_p[y_b] \times \{b\}$. Thus $\langle z, b\rangle \in W^2[\langle x, a\rangle]$ and, by (III), $\langle z, b\rangle \notin \hat{U}^{n+1}[\langle x, a\rangle] = \hat{U}^{(n+1)+}[\langle x, a\rangle]$, as required.

PROPOSITION 2. *Suppose $\langle x, a\rangle \in Z$ has a neighbourhood $\hat{G} = \{\langle x, a\rangle\} \cup \bigcup_{i=1}^{\infty} G_i \times \{b_i\}$, where the G_i are clopen in Y and the b_i converge in $\mathbf{R}$ to a. If $\langle x, a\rangle$ is the only point of Z in $Y \times \{a\}$, then $\hat{G}$ is clopen in $\hat{Y}$.*

LEMMA 3. *Suppose* (a) *each V_n is a clopen neighbournet on Y*; (b) *U^k is a clopen neighbournet for each $k \in \mathbf{N}$; and* (c) *$U^k \circ V_n$ is a clopen neighbournet for each k, $n \in \mathbf{N}$. If $\hat{Y}$ is constructed so that*

(IV) *for each $a \in A$ there is at most one point of Z in $Y \times \{a\}$; and*

(V) *for each $\langle x, a\rangle \in Z$, $S(x, a)$ is a sequence $\{\langle y_i, b_i\rangle: i \in \mathbf{N}\}$ where the b_i converge in $\mathbf{R}$ to a;*

then (â) *each $\hat{V}_n$ is a clopen neighbournet on $\hat{Y}$*; ($\hat{\text{b}}$) *$\hat{U}^k$ is a clopen neighbournet for each $k \in \mathbf{N}$; and* (ĉ) *$\hat{U}^k \circ \hat{V}_n$ is a clopen neighbournet for each $k, n \in \mathbf{N}$.*

PROOF. It will suffice to show for each $\langle x, a\rangle \in Z$ that $\hat{V}_n[\langle x, a\rangle]$, $\hat{U}^k[\langle x, a\rangle]$ and $\hat{U}^k \circ \hat{V}_n[\langle x,a\rangle]$ are clopen. Suppose $\langle x, a\rangle \in Z_p$. By (V), let $S(x, a) = \{\langle y_i, b_i\rangle: i \in \mathbf{N}\}$. Then

$$\hat{V}_n[\langle x, a\rangle] = \{\langle x, a\rangle\} \cup \bigcup\{V_n[y_i] \times \{b_i\}: |b_i - a| < 2^{-n}\},$$

$$\hat{U}^k[\langle x, a\rangle] = \{\langle x, a\rangle\} \cup \bigcup\{U^{k-1} \circ V_p[y_i] \times \{b_i\}: |b_i - a| < 2^{-p}\},$$

$$\hat{U}^k \circ \hat{V}_n[\langle x, a\rangle] = \{\langle x, a\rangle\} \cup \bigcup\{U^k \circ V_n[y_i] \times \{b_i\}: |b_i - a| < 2^{-n}\}$$
$$\cup \bigcup\{U^{k-1} \circ V_p[y_j] \times \{b_j\}: |b_j - a| < 2^{-p}\}.$$

The required result now follows from Proposition 2, using (IV) and the assumptions (a) and (c). (Note that if $k - 1 = 0$ then $U^{k-1} \circ V_p = V_p$, and so we would use (a) instead of (c) to guarantee that $U^{k-1} \circ V_p[y]$ was clopen.)

Now let us show that conditions (I) through (V) from Lemmas 1 and 3 may be met by suitably constructing $\hat{Y}$ from a space Y of cardinality at most $\mathbf{c}$.

First, we may partition $\mathbf{R}$ into sets A and B, where A has cardinality $\mathbf{c}$ on every open interval of $\mathbf{R}$ and B is dense and Baire in $\mathbf{R}$.

If the cardinality of Y is no more than $\mathbf{c}$, there will also be no more than $\mathbf{c}$ countable subsets E of $Y \times B$. Then by a straightforward transfinite induction, choose for each countable $E \subseteq Y \times B$ whose canonical projection onto B is dense in some interval (c_1, c_2) in $\mathbf{R}$ and for each $p \in \mathbf{N}$, a *distinct* real number $a_{Ep} \in A \cap (c_1, c_2)$. Choose an arbitrary $x_{Ep} \in Y$ and let $S(x_{Ep}, a_{Ep})$ be any sequence of points $\langle y_i, b_i\rangle$ in E where the b_i are distinct and converge in $\mathbf{R}$ to a_{Ep}.

Let Z consist of all points $\langle x_{Ep}, a_{Ep}\rangle$; and partition A into sets A_p ($p \in \mathbf{N}$) so that $a_{Ep} \in A_p$. Then conditions (I) through (V) are met (note that it is sufficient to prove (II) for countable sets E).

4. The counterexample. To complete the construction of the counterexample $\tilde{X}$, all that remains to be done is to provide a suitable space X_0 to start off the induction. For this purpose we will choose the convergent sequence space $\{2^{-k}: k \in \mathbf{N}\} \cup \{0\}$. Observe that X_0 is not 0^+-pretransitive, and in fact U_0^{0+} will not be a normal neighbournet no matter what U_0 is. To satisfy the inductive assumptions (a), (b) and (c) of Lemma 3 we may define the neighbounet U_0 and the normal basic sequence $\langle V_n: n \in \mathbf{N}\rangle$ so that $U_0[x] = X_0$ and $V_n[2^{-k}] = \{2^{-k}\}$; $V_n[0] = \{2^{-k}: k > n\} \cup \{0\}$.

The counterexample $\tilde{X}$ thereby produced will have the following properties. Note that the construction of $\tilde{Y}$ from Y and the construction of $\tilde{X}$ from X both preserve scatteredness, and that both increase the Cantor-Bendixson rank of the space by 1. Therefore $\tilde{X}$ will be scattered, and the Cantor-Bendixson rank of $\tilde{X}$ will be $\omega + 1$. Consequently, $\tilde{X}$ will be transitive (by transfinite induction and **[FL**, 6.16 and 6.17]), hereditarily weakly θ-refinable, and quasi-developable. Junnila has shown in **[J2]** that developable γ-spaces are quasi-metrizable; this demonstrates that developable cannot be weakened to quasi-developable.

We remark in passing that, with a modified construction of $\hat{Y}$, a counterexample $\tilde{X}$ can be constructed which has the above properties and which is in addition submetrizable—that is, it has normal G_δ-diagonal sequence.

Finally, we observe that the construction given in **[F]** and used here cannot produce a normal counterexample to the γ-space conjecture. In particular, if X is any T_1 space containing at least 2 points then $\tilde{X}$ will not be normal. For example, if X is the 2-point discrete space $\{0, 1\}$ then $\tilde{X}$ will consist of levels 1 through ω inclusive of a Cantor tree with the tree topology, a nonnormal space. To see this for a larger space X, consider a 2-point subset of X and the Cantor tree it generates in $\tilde{X}$. This raises the question: Are *normal* γ-spaces quasi-metrizable?

ADDED IN PROOF. The answer to the last question is *no*; there exists a paracompact counterexample.

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