SPACES WITH NONCOINCIDING DIMENSIONS

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ABSTRACT. For any given nonnegative integers l, m, n with $\max\{l, m\} \leq n$ and $n = 0$ if $m = 0$, we construct a normal, Hausdorff and separable space X with $\text{ind } X = l$, $\dim X = m$ and $\text{Ind } X = n$. We also construct a space X_n with $\dim X_n = 1$ and $\text{ind } X_n = \text{Ind } X_n = n$ which is the limit space of an inverse limit sequence of compact, Hausdorff and separable spaces all of whose dimensions are one.

1. Introduction. In §2, for any given nonnegative integers l, m, n with $\max\{l, m\} \leq n$ and $n = 0$ if $m = 0$, we construct a normal, Hausdorff and separable space X with $\text{ind } X = l$, $\dim X = m$ and $\text{Ind } X = n$. In particular, we establish for $m \leq n$ the existence of a normal Hausdorff space X with $\text{ind } X = 0$, $\dim X = m$ and $\text{Ind } X = n$. Apparently, the only existing example of this sort is that of Nagami [11] for $m = 1$ and $n = 2$. Furthermore, there is no example of a space X in the literature known to satisfy, for example, $\dim X = 1$, $\text{ind } X = 2$ and $\text{Ind } X = 4$, although Filippov [8] constructs a compact Hausdorff space X with $\dim X = 1$, $\text{ind } X = i$ and $\text{Ind } X = 2i - 1$ for any given positive integer i .

Assuming CH, the Continuum Hypothesis, our space X can be constructed to be additionally perfectly normal. Several examples of perfectly normal separable spaces with noncoinciding dimensions have been constructed using CH (see e.g. [7, 9, 15]). However, the only known perfectly normal spaces with noncoinciding dimensions that were constructed using no set-theoretical assumptions beyond ZFC are not separable [12, 14].

In §3, given a positive integer n , we construct an inverse limit sequence of compact, Hausdorff and separable spaces each of which has all dimensions equal to one while the limit space X_n of the sequence satisfies $\dim X_n = 1$ and $\text{ind } X_n = \text{Ind } X_n = n$. For $n = 2$, we constructed such an example in [2]. Again assuming CH, all the spaces of the sequence as well as the limit space X_n can be constructed to be additionally perfectly normal. This is in sharp contrast with the inverse limit theorem for covering dimension which holds not only for arbitrary inverse systems of compact spaces, but also for sequences of perfectly normal spaces [1].

As regards method of construction, in both cases we construct new topologies from old using a mixture of the techniques of replacing a point by a space (cf. [5, 7, 9]) and the techniques of assigning limit points to enough sequences so as not to deviate too much from the original topology (see [4, 10, 13, 15]).

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In this paper, all spaces are at least Tychonoff. N denotes the set of positive integers, C the Cantor set, $I = [0, 1]$ the unit interval, I^n the n -dimensional unit cube, $|X|$ the cardinality of a set X , c the cardinality of the continuum and $\omega(c)$ the first ordinal of cardinality c .

For the standard results in Dimension Theory we refer to [16, 17].

2. Construction of normal spaces with dimensions given integers.

THEOREM 1. *Let X be a normal, Hausdorff and separable space X with $\dim X \geq 1$. Then there exists a normal, Hausdorff and separable space $Y = Y(X)$ such that*

- (i) $\text{ind } Y = \text{ind } X$,
- (ii) $\dim Y = \dim X$, and
- (iii) $\text{Ind } Y = \text{Ind } X + 1$.

PROOF. Let D be a separable completely metrizable one-dimensional space that contains two disjoint closed subsets E_0, F_0 which cannot be separated by countable closed set. An example of such a space is $C \times I$. Let Q, P be countable dense subsets of D, X , respectively. Let $\{(S_\alpha, T_\alpha): \alpha < \omega(c)\}$ be the collection of all sequences of D with $\bar{S}_\alpha \cap \bar{T}_\alpha$ uncountable. Note that, in fact, $|\bar{S}_\alpha \cap \bar{T}_\alpha| = c$ since D is separable and completely metrizable. For each $\alpha < \omega(c)$, pick by transfinite induction a point x_α and a sequence $\{x_{\alpha n}\}$ converging to x_α so that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\{x_{\alpha n}\}$ contains infinitely many points from each of S_α, T_α and Q and $x_{\alpha n} \triangleleft x_\alpha$ for each n in N , where $\triangleleft$ is some well ordering on D of the same type as $\omega(c)$ such that $q \triangleleft x$ whenever $q \in Q$ and $x \notin Q$. Let $f_\alpha: \{x_{\alpha n}\} \rightarrow P$ be any function such that, for every x in P , $f_\alpha^{-1}(x)$ consists of infinitely many points from each of S_α, T_α and Q . Let $A = \{x_\alpha: \alpha < \omega(c)\}$. For each $x \notin A$, we also fix a sequence $\{x_n\}$ in Q converging to x .

We define a new topology on D by transfinite induction as follows. We let every point of Q be isolated and, assuming we have defined basic open neighbourhoods of all points $\triangleleft x$, we declare basic open neighbourhoods of x sets of the form $\{x\} \cup W$ where W is an open neighbourhood of all but finitely many points of $\{x_n\}$. D^* will denote D with this new topology, which is clearly separable and finer than the standard topology on D . Moreover, each point of D^* has countable basic open neighbourhoods which are closed in D and hence in D^* . Thus $\text{ind } D^* = 0$ and D^* is Tychonoff.

We now let

$$Y = D \times \{p\} \cup A \times X,$$

where p is a fixed point of P , with topology defined as follows.

(a) If $x \notin A$, (x, p) has basic open neighbourhoods sets of the form $\pi^{-1}(U)$ where $\pi: Y \rightarrow D$ is the coordinate projection and U is an open neighbourhood of x in D^* .

(b) $(x_\alpha, y) \in A \times X$ has basic open neighbourhoods sets of the form

$$O(x_\alpha, U, V) = \{x_\alpha\} \times V \cup \pi^{-1}(U),$$

where V is an open neighbourhood of y in X and U is an open neighbourhood in D^* of all but finitely many points of $f_\alpha^{-1}(V)$.

Clearly Y is a T_1 separable space and π is continuous both as a function onto D and D^* . Furthermore,

(1) $\{x_\alpha\} \times X \subset \bar{Z}$ whenever $\pi(Z)$ contains S_α or T_α .

(2) If E, F are disjoint closed sets of Y , then the set $\overline{\pi(E)} \cap \overline{\pi(F)}$ is countable in D and hence in D^* . Otherwise, for some $\alpha < \omega(c)$, $S_\alpha \subset \pi(E)$ and $T_\alpha \subset \pi(F)$ and hence, by (1), $\{x_\alpha\} \times X \subset E \cap F$.

LEMMA 1. Y is normal.

PROOF. Let E, F be disjoint closed sets of Y . The countable set $\overline{\pi(E)} \cap \overline{\pi(F)}$ is contained in a countable cozero set Z of the locally countable space D^* . Further, since $\text{ind } D^* = 0$, Z is Lindelöf with $\dim Z = 0$, and we can insert a clopen set of D^* between $\overline{\pi(E)} \cap \overline{\pi(F)}$ and Z . To prove the lemma, it clearly suffices to show that $\pi^{-1}(Z)$ is normal for all basic neighbourhoods of D^* , which are countable and clopen. If Z is such a neighbourhood of $x \notin A$, then $\pi^{-1}(Z)$ is homeomorphic with Z which is normal. Let us assume, therefore, that Z is a basic neighbourhood of x_α and E, F are disjoint closed sets of $\pi^{-1}(Z)$. We may assume further that $x \triangleleft x_\alpha$ for $x \in Z - \{x_\alpha\}$. Let V_1, V_2 be disjoint open sets of X with $(\{x_\alpha\} \times X \cap E) \subset \{x_\alpha\} \times V_1$ and $(\{x_\alpha\} \times X \cap F) \subset \{x_\alpha\} \times V_2$. If $K = \overline{\pi(E)} \cap \overline{\pi(F)} - \{x_\alpha\}$, there exist disjoint clopen sets Z_1, Z_2 of $Z - \{x_\alpha\}$ containing K and $\{x_{\alpha n}\} - K$, respectively. Then $\pi^{-1}(\{x_\alpha\} \cup Z_2)$ is in fact clopen in $\pi^{-1}(Z)$ since no point of $\{x_\alpha\} \times X$ is a limit point of $\pi^{-1}(K)$. Since $x \triangleleft x_\alpha$ for $x \in Z_1$, by transfinite induction, it suffices to prove normality in the case $Z = \{x_\alpha\} \cup Z_2$, i.e. when $\overline{\pi(E)} \cap \overline{\pi(F)} = \{x_\alpha\}$. Let U_1, U_2 be disjoint clopen sets of Z_2 containing, respectively, $(f_\alpha^{-1}(V_1) - \pi(F)) \cup (\pi(E) - \{x_\alpha\})$ and the union of its complement in $\{x_{\alpha n}\}$ with $\pi(F) - \{x_\alpha\}$. Then $O(x_\alpha, U_1, V_1), O(x_\alpha, U_2, V_2)$ are disjoint open sets of Y containing, respectively, E, F , which concludes the proof of Lemma 1.

We note that, in fact, $\text{bd } O(x_\alpha, U_1, V_1) = \{x_\alpha\} \times \text{bd } V_1$, which readily implies the following two results.

LEMMA 2. $\text{ind } Y = \text{ind } X$.

LEMMA 3. If Z is a countable closed or open subspace of D^* , then $\text{Ind } \pi^{-1}(Z) \leq \text{Ind } X$.

LEMMA 4. $\dim Y = \dim X$.

PROOF. Set $n = \dim X \geq 1$. Let $(E_1, F_1), \dots, (E_{n+1}, F_{n+1})$ be pairs of disjoint closed subsets of Y . As in Lemma 1, there exists a countable clopen set Z of D^* which contains the closed sets $\overline{\pi(E_i)} \cap \overline{\pi(F_i)}$ of D , $i = 1, \dots, n+1$. Since $n \geq 1$ and $\dim D - Z \leq \dim D \leq 1$, $\overline{\pi(E_i)} - Z, \overline{\pi(F_i)} - Z$, $i = 1, \dots, n+1$, can be separated in $D - Z$ by closed sets K_i such that $\bigcap_{i=1}^{n+1} K_i = \emptyset$. Further, the normal space $\pi^{-1}(Z)$ is the union of a countable collection of closed subsets each of which is a singleton or a copy of X . Hence $\dim \pi^{-1}(Z) \leq n$ and $E_i \cap \pi^{-1}(Z), F_i \cap \pi^{-1}(Z)$ can be separated in $\pi^{-1}(Z)$ by disjoint closed sets L_i , $i = 1, \dots, n+1$, with $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Let $M_i = \pi^{-1}(K_i) \cup L_i$. This is clearly a closed partition of the pair E_i, F_i in Y and $\bigcap_{i=1}^{n+1} M_i = \emptyset$. Hence $\dim Y \leq \dim X$. That $\dim Y \geq \dim X$ follows from the fact that Y contains a copy of X .

LEMMA 5. Let L be a closed subset of Y with $\pi(L)$ a zero-dimensional subset of D . Then $\text{Ind } L \leq \text{Ind } X$.

PROOF. Let E, F be disjoint closed sets of L . Then the closed set $\overline{\pi(E)} \cap \overline{\pi(F)}$ of D is contained in a countable clopen subset Z of D^* . Now $\pi(E) - Z, \pi(F) - Z$ are separated by the empty set in the zero-dimensional space $\pi(L) - Z$. Hence E, F can be separated by the empty set in $L - \pi^{-1}(Z)$ and so any partition of these sets in $\pi^{-1}(Z)$ separates them in L as well. By Lemma 3, there is such a partition with $\text{Ind} \leq \text{Ind } X - 1$. Hence $\text{Ind } L \leq \text{Ind } X$.

LEMMA 6. $\text{Ind } Y \leq \text{Ind } X + 1$.

PROOF. Any two disjoint closed sets E, F can be separated by a set of the form $K = \pi^{-1}(M) \cup L$ where $L \subset \pi^{-1}(Z)$ for some countable clopen set Z of D^* , $\text{Ind } L \leq \text{Ind } X - 1$, and M is a zero-dimensional closed subset of the one-dimensional space $D - Z$. By Lemma 5, $\text{Ind } \pi^{-1}(M) \leq \text{Ind } X$. Hence, since $\pi^{-1}(M), L$ are clopen in K , $\text{Ind } K \leq \text{Ind } X$ and so $\text{Ind } Y \leq \text{Ind } X + 1$.

LEMMA 7. $\text{Ind } Y > \text{Ind } X$.

PROOF. Let E_0, F_0 be disjoint closed sets of D which cannot be separated by a countable set. Let U, V be open sets of D with $E_0 \subset U, F_0 \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$. Let E, F be closed sets of Y with $Y = E \cup F, \pi^{-1}(\overline{U}) \subset E$ and $\pi^{-1}(\overline{V}) \subset F$. It suffices to show $\text{Ind}(E \cap F) \geq \text{Ind } X$. But $\overline{\pi(E)} \cap \overline{\pi(F)}$ is a partition in D between E_0 and F_0 . Hence for some $\alpha < \omega(c)$, S_α, T_α are dense subsets of $\pi(E), \pi(F)$, respectively. Hence, by (1), $E \cap F$ contains $\{x_\alpha\} \times X$ and so $\text{Ind}(E \cap F) \geq \text{Ind } X$.

Lemmas 6 and 7 imply that $\text{Ind } Y = \text{Ind } X + 1$, which concludes the proof of Theorem 1.

REMARK 1 (CH). If X is perfectly normal, Y can be made perfectly normal.

PROOF. Let $\{R_\alpha: \alpha < \omega(c)\}$ be the set of all sequences of D with $R_0 = Q$. Then assuming CH, $L_x = \{\alpha: x_\alpha \triangleleft x\}$ is countable for each x in D . Thus we can choose the sequence $\{x_n\}$ with the additional property that it contains infinitely many points from each R_α with $\alpha \in L_x, y \triangleleft x$ for each y in R_α and $x \in \overline{R}_\alpha$. Now for each closed set E of Y ,

(3) $\overline{\pi(E)} - \pi(E)$ in D is countable.

PROOF. Otherwise, let R_α be a dense countable subset of $\pi(E)$ and choose x in $\overline{\pi(E)} - \pi(E)$ such that $x_\alpha \triangleleft x$ and $y \triangleleft x$ for each $y \in R_\alpha$. Then $\pi^{-1}(x)$ is contained in the closed set E , contradicting the fact that $x \notin \pi(E)$.

Also, if G is open in Y ,

(4) $\pi(G) - \pi^\#(G)$ is countable, where $\pi^\#(G) = D - \pi(Y - G)$.

PROOF. If not, for some $\alpha < \omega(c)$, R_α is a dense subset of $\pi(G) - \pi^\#(G)$. Let x be a point of this set such that $x_\alpha \triangleleft x$ and $y \triangleleft x$ for each $y \in R_\alpha$. The definition of the topology of Y assures that an open set $O(x, U, V)$ inside G contains $\pi^{-1}(y)$ for infinitely many points y of R_α , contradicting the fact that R_α is a subset of $\pi(G) - \pi^\#(G)$.

It follows from (3) and (4) that

$$G = \pi^{-1}(D - \overline{\pi(Y - G)}) \cup (G \cap \pi^{-1}Z)$$

where Z is a countable subset of D . The conclusion that Y is perfectly normal if X is is now immediate.

COROLLARY 1. *Let l, m, n be nonnegative integers with $l \leq m \leq n$ and $m \geq 1$. Then there exists a separable, Hausdorff and normal space Z with $\text{ind } Z = l, \text{dim } Z = m$ and $\text{Ind } Z = n$.*

PROOF. Let X be the disjoint union of I^l with a normal T_2 and separable space with $\text{ind} = 0$ and $\text{dim} = \text{Ind} = m$ (e.g. [3]). Clearly, $\text{ind } X = l$ and $\text{dim } X = \text{Ind } X = m$. Finally, apply Theorem 1 to get spaces $X_1 = Y(X), X_2 = Y(X_1), \dots, Z = X_{n-m} = Y(X_{n-m-1})$. Clearly, Z has the stated properties.

COROLLARY 2 (CH). *Let l, m, n be nonnegative integers with $l \leq m \leq n$ and $m \geq 1$. Then there exists a perfectly normal, separable and Hausdorff space Z with $\text{ind } Z = l, \text{dim } Z = m$ and $\text{Ind } Z = n$.*

PROOF. A modification of the construction in [3] as in Remark 1 will produce a perfectly normal, separable and Hausdorff space T with $\text{ind } T = 0$ and $\text{dim } T = \text{Ind } T = m$.

We note that Fedorčuk [7] also constructs spaces with properties like those of T as well as spaces with the properties of Z of Corollary 2. His construction uses CH too.

COROLLARY 3. *Let l, m, n be positive integers with $l \leq m \leq n$. Then there exists a normal, separable and Hausdorff space with $\text{dim } Z = l, \text{ind } Z = m$ and $\text{Ind } Z = n$.*

PROOF. Let X be the disjoint union of I^l and a compact T_2 and separable space with $\text{dim} = 1$ and $\text{ind} = \text{Ind} = m$ (see [9] or §3 of this paper). If $X_1 = Y(X), X_2 = Y(X_2), \dots, Z = X_{n-m} = Y(X_{n-m-1})$, then Z has the stated properties.

COROLLARY 4 (CH). *Let l, m, n be positive integers with $l \leq m \leq n$. Then there is a perfectly normal, Hausdorff and separable space Z with $\text{dim } Z = l, \text{ind } Z = m$ and $\text{Ind } Z = n$.*

PROOF. There exist, under CH, compact T_2 , perfectly normal and separable spaces with $\text{dim} = 1$ and $\text{ind} = \text{Ind} = m$ (see [7, 9] or §3 of this paper).

3. Dimension and inverse limit sequences.

THEOREM 2. *For each integer $n \geq 1$, there is an inverse limit sequence of compact, Hausdorff and separable spaces each of which has $\text{dim} = \text{ind} = \text{Ind} = 1$ with limit space X_n satisfying $\text{dim } X_n = 1$ while $\text{ind } X_n = \text{Ind } X_n = n$.*

PROOF. We perform a construction similar to that of the previous section. Here D will be the unit interval I together with the union of a collection $\{I_n; n \in N\}$ of copies of I such that I_{2n-1}, I_{2n} have endpoints $(0, 1/(2n-1)), (1, 1/2n)$ and $(1, 1/2n), (0, 1/(2n+1))$, respectively. $\{(S_\alpha, T_\alpha); \alpha < \omega(c)\}$ is the collection of all sequences of I with $|\bar{S}_\alpha \cap \bar{T}_\alpha| = c$ and x_α in $\bar{S}_\alpha \cap \bar{T}_\alpha$ is chosen so that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. For each x in I , we choose a sequence $\{x_n\}$ converging to x and such that if $x = x_\alpha$ for some $\alpha < \omega(c)$, then $x_{an} \in S_\alpha$ if n is odd and $x_{an} \in T_\alpha$ if n is even. Let $Q_1, Q_2, Q_3, \dots$ be countable dense subsets of I contained respectively in $Q(\sqrt{2}) - Q$,

$Q(\sqrt{2}, \sqrt{3}) - Q(\sqrt{2})$, $Q(\sqrt{2}, \sqrt{3}, \sqrt{5}) - Q(\sqrt{2}, \sqrt{3})$, ..., where $Q(a_1, a_2, \dots, a_k)$ denotes the field extension of Q , the field of rational numbers, generated by $a_1, a_2, \dots, a_k$. It is important that if $q \in Q_m$, $r \in Q_n$ and $m \neq n$, the lines $x + y = q$ and $x - y = r$ do not meet on D . For $0 < x < 1$, let $h_x: I - \{x\} \rightarrow (x, 1]$ be a continuous extension of the identity on $(x, 1]$ that maps $[0, x)$ homeomorphically onto $(x, 1]$ and takes $Q_n - \{x\}$ to $Q_n \cap (x, 1]$ for each n in N ; let $g_x: (x, 1] \rightarrow D - I$ be a homeomorphism and define f_x to be $g_x \circ h_x$. For $x = 0, 1$, we let $f_x: I - \{x\} \rightarrow D - I$ be any homeomorphism. For all x in I , we can assume that, for each n in N , f_x maps only points of Q_n to points of intersection of lines of the form $x + y = q$, $x - y = r$ with q and r in Q_n , and that the distance between $f_x(x_n)$ and $f_x(x_{n+1})$ tends to zero. The following properties of f_x are immediate:

(5) $f_x^{-1}(V)$ contains infinitely many points of $\{x_n\}$ and, if $x = x_\alpha$, it contains infinitely many points from both S_α and T_α , whenever V is an open set of D with $V \cap I \neq \emptyset$.

(6) $\{x\} \cup f_x^{-1}(V)$ is open in I whenever V is an open set of D that contains all but finitely many I_n 's.

Next we consider the category $\mathcal{C}$ whose objects are pairs (X, π) where X is a compact, T_2 and separable space and $\pi: X \rightarrow I$ is a continuous surjection such that, for each x in I ,

(7) For every finite open cover $\{V_1, V_2, \dots, V_k\}$ of $\pi^{-1}(x)$, $\{x\} \cup \bigcup_{i=1}^k \pi^\#(V_i)$ is open in I , where $\pi^\#(V) = I - \pi(X - V)$, i.e. π is fully closed [6].

(8) Every point $\pi^{-1}(x)$ has basic neighbourhoods V such that $\pi(V) - \{x\}$ is open in I and its closure contains x . For each such neighbourhood which in the sequel is referred to as special, V^* denotes the set consisting of the intersection of D with the perimeters of all squares one diameter of which has endpoints $y, h_x(y)$ with either y or $h_x(y)$ in $\pi(V) - \{x\}$, where $h_0(y) = -y$ and $h_1(y) = 1 + y$.

(9) Any two distinct points of $\pi^{-1}(x)$ have special neighbourhoods V_1, V_2 such that $V_1^* \cap V_2^* = \emptyset$.

A morphism $(X, \pi) \rightarrow (Y, \sigma)$ in $\mathcal{C}$ is a map $\tau: X \rightarrow Y$ such that $\pi = \sigma \circ \tau$. It will prove convenient to occasionally denote the object (X, π) of $\mathcal{C}$ more simply by X .

Given (X, π) in $\mathcal{C}$ and $A \subset I$, we define a topology on

$$Y(A) = Y(A, (X, \pi)) = I \times \{q\} \cup A \times X$$

where q is a fixed point of X , as follows;

(a) For $x \notin A$, (x, q) has basic open neighbourhoods of the form $\sigma^{-1}(U)$ where U is an open neighbourhood of x in I and $\sigma: Y(A) \rightarrow I$ is the natural projection.

(b) For $x \in A$, (x, y) has basic open neighbourhoods of the form

$$O(x, U, V) = \{x\} \times V \cup \sigma^{-1}(f_x^{-1}(V^*) \cap U)$$

where U is an open neighbourhood of x in I and V is a special neighbourhood of y in X .

It can be verified that $(Y(A), \sigma)$ is in $\mathcal{C}$ and $\{x_\alpha\} \times X$ is contained in the closure of any subset Z of $Y(A)$ with S_α or $T_\alpha \subset \sigma(Z)$. The fact that σ is fully closed, for

instance, follows from (6) and the fact that π is fully closed. Furthermore, if to a morphism $\pi: X \rightarrow Z$ in $\mathcal{C}$, we associate $\tau^* = \tau^*(A): Y(A, X) \rightarrow Y(A, Z)$ defined by $\tau^*(x, y) = (x, \tau(y))$, we see that we have a functor $Y: \mathcal{C} \rightarrow \mathcal{C}$.

LEMMA 8. $\text{ind } Y(I) \geq \text{ind } X + 1$ and $\text{Ind } Y(I) \geq \text{Ind } X + 1$.

PROOF. Let E, F be nontrivial regular closed subsets of $Y(I)$ with $Y(I) = E \cup F$. If $\sigma(E) \cap \sigma(F)$ is uncountable, then for some $\alpha < \omega(c)$, $S_\alpha \subset \sigma(E)$, $T_\alpha \subset \sigma(F)$ and $\{x_\alpha\} \times X \subset E \cap F$. If $\sigma(E) \cap \sigma(F)$ is countable, it contains an isolated point x . Then the regularity of E and F assures that $\{x\} \times X$ is contained in $E \cap F$. This proves the result.

To compute the inductive dimensions of $Y(A)$, it is convenient to assume that X satisfies the following:

(10) For n in N sufficiently large, every point of $\pi^{-1}(x)$ has arbitrarily small special neighbourhoods V with $\pi(\text{bd } V) - \{x\} \subset Q_n$, $\text{ind } \text{bd } V < \text{ind } X$ and $\text{Ind } \text{bd } V < \text{Ind } X$.

And

(11) If B is a subset of I of the form $Q_1 \cup \dots \cup Q_n \cup \{x_1, \dots, x_n\}$, then $\text{ind } \pi^{-1}(B) < \text{ind } X$ and $\text{Ind } \pi^{-1}(B) < \text{Ind } X$.

LEMMA 9. If X satisfies (10) and (11), then $\text{ind } Y(I) = \text{ind } X + 1$, $\text{Ind } Y(I) = \text{Ind } X + 1$ and $Y(I)$ satisfies (10) and (11).

PROOF. If V is as in (10) and U is an open interval around x with endpoints in Q_n , then the construction of f_x assures that $O(x, U, V)$ has boundary in $\sigma^{-1}(Q_n \cup \{x\})$. Now consider a subset $B = Q_1 \cup \dots \cup Q_n \cup \{x_1, x_2, \dots, x_n\}$ of I . Let (x, y) be in $\sigma^{-1}(B)$ and V a special neighbourhood of y in X with $\text{bd } \pi(V) - \{\pi(y)\} \subset Q_{n+1}$ and U on open interval with endpoints in Q_{n+1} containing x but no point from $\{x_1, x_2, \dots, x_n\} - \{x\}$. Then the boundary of $O(x, U, V)$ in $\sigma^{-1}(B)$ is $\{x\} \times \text{bd } V$. Hence, $\text{ind } \sigma^{-1}(B) \leq \text{ind } X$ and $\text{ind } Y(I) \leq \text{ind } X + 1$. Thus, in view of Lemma 8, $\text{ind } Y(I) = \text{ind } X + 1$ and $Y(I)$ satisfies (10). The rest of the lemma follows from the fact that X satisfied (11) and for every closed set F inside an open set G of X there are special neighbourhoods $V_1, \dots, V_k$ satisfying (10) and $F \subset \bigcup_{i=1}^k V_i \subset G$.

In the sequel, for each $n \in N$, A_n denotes the set $I - \bigcup_{i=n}^\infty Q_i$.

LEMMA 10. If X satisfies (10) and $\text{ind } X = 1$, then the same holds for $Y(A_n)$.

PROOF. Let V be a special neighbourhood of y in X with $\text{ind } \text{bd } V \leq 0$ and $\pi(\text{bd } V - \{y\})$ a subset of Q_{n+1} . Let U be an open interval containing x with endpoints in Q_{n+1} . Then $\text{bd } O(x, U, V)$ consists of $\{x\} \times \text{bd } V$ together with a subset of $\sigma^{-1}(Q_{n+1})$, which is homeomorphic with its projection under σ into I . Now the sum theorem for covering dimension assures that $\text{bd } O(x, U, V)$ is zero-dimensional in every sense.

To conclude the proof of Theorem 2, two remarks are needed. First, if $A \subset B \subset I$ then there is a map $Y(B, X) \rightarrow Y(A, X)$ in $\mathcal{C}$ that sends (x, y) to (x, q) for

$x \in B - A$ and leaves all other points fixed. Moreover, if $X \rightarrow Z$ is a map in $\mathcal{C}$, we have a map $Y(B, X) \rightarrow Y(A, Z)$ in $\mathcal{C}$ which is the composite of $Y(B, X) \rightarrow Y(A, X)$ with the map $Y(A, X) \rightarrow Y(A, Z)$ considered before. Second, if

$$Z \rightarrow \cdots \rightarrow Z_n \rightarrow Z_{n-1} \rightarrow \cdots \rightarrow Z_1$$

is an inverse limit sequence in $\mathcal{C}$ with limit Z , then

$$Y(I, Z) \rightarrow \cdots \rightarrow Y(A_n, Z_n) \rightarrow Y(A_{n-1}, Z_{n-1}) \rightarrow \cdots \rightarrow Y(A_1, Z_1)$$

is an inverse limit sequence in $\mathcal{C}$ with limit $Y(I, Z)$.

Thus by repeated application of the functor Y we obtain the following sequences and limit spaces in $\mathcal{C}$:

$$\begin{aligned} X_2 &= Y(I, I) \rightarrow \cdots \rightarrow Y(A_m, I) \rightarrow Y(A_{m-1}, I) \rightarrow \cdots \rightarrow Y(A_1, I), \\ X_3 &= Y(I, X_2) \rightarrow \cdots \rightarrow Y(A_m, Z_{m,2}) \rightarrow Y(A_{m-1}, Z_{m-1,2}) \\ &\rightarrow \cdots \rightarrow Y(A_1, Z_{1,2}), \\ X_n &= Y(I, X_{n-1}) \rightarrow \cdots \rightarrow Y(A_m, Z_{m,n-1}) \rightarrow Y(A_{m-1}, Z_{m-1,n-1}) \\ &\rightarrow \cdots \rightarrow Y(A_1, Z_{1,n-1}), \end{aligned}$$

where $Z_{m,k} = Y(A_m, Z_{m,k-1})$ for $k > 1$ and $Z_{m,1} = I$.

Repeated application of Lemma 9 shows that $\text{ind } X_n = \text{Ind } X_n = n$, while Lemma 10 assures that $\text{ind } Z_{n,k} = \text{Ind } Z_{n,k} = 1$. That $\dim X_n = 1$ follows from the fact that X_n is the inverse limit of compact spaces with covering dimension 1.

REMARK 2. As in Theorem 1, assuming CH, the spaces of Theorem 2 can be constructed to be additionally perfectly normal.

REMARK 3. It would be interesting to know whether Remarks 1 and 2 as well as Corollaries 2 and 4 remain valid without CH. Indeed the following open problems pertaining to spaces not all of whose dimensions coincide present themselves. The last three of these are open even assuming CH. Roy's example of a metric space with $\text{ind} = 0$ and $\dim = 1$ is the only known partial solution of (6). As for (4), apart from Filippov's examples in [8] various other examples from the same period exist that provide partial solutions, but in its general form, we believe, the problem is as yet unresolved.

1. Given nonnegative integers l, m, n with $l \leq m \leq n$ and $m \geq 1$, construct a normal Hausdorff space X such that $\text{ind } X = l$, $\dim X = m$, $\text{Ind } X = n$ and X is additionally perfectly normal or completely normal.

2. Given positive integers l, m, n with $l \leq m \leq n$, construct a normal Hausdorff space X such that $\dim X = l$, $\text{ind } X = m$, $\text{Ind } X = n$ and X is additionally perfectly normal or completely normal.

3. Construct a separable perfectly normal or ever completely normal space X with $\text{ind } X < \dim X$.

4. Given positive integers l, m, n with $l \leq m \leq n$, construct a compact Hausdorff space X with $\dim X = l$, $\text{ind } X = m$ and $\text{Ind } X = n$.

5. The same as (4) with X being additionally perfectly normal or completely normal.

6. Given nonnegative integers l and m with $l \leq m$, construct a metric space X with $\text{ind } X = l$ and $\dim X = n$.

REFERENCES

1. M. G. Charalambous, *The dimension of inverse limits*, Proc. Amer. Math. Soc. **58** (1976), 289–295.
2. ———, *Inductive dimension and inverse sequences of compact spaces*, Proc. Amer. Math. Soc. **81** (1981), 482–484.
3. ———, *The dimension of inverse limit and N -compact spaces*, Proc. Amer. Math. Soc. **85** (1982), 648–652.
4. E. Van Douwen, *A technique for constructing honest locally compact submetrizable spaces*, Preprint.
5. V. V. Fedorčuk, *Bcompacta with non-coinciding dimensionalities*, Soviet Math. Dokl. **9** (1968), 1148–1150.
6. ———, *Mappings that do not reduce dimension*, Soviet Math. Dokl. **10** (1969), 314–317.
7. ———, *On the dimension of hereditarily normal spaces*, Proc. London Math. Soc. (3) **36** (1978), 163–175.
8. V. V. Filippov, *On bcompacta with non-coinciding inductive dimensions*, Soviet Math. Dokl. **11** (1970), 635–638.
9. V. V. Filippov, *On compacta with unequal dimensions ind and dim* , Soviet Math. Dokl. **11** (1970), 687–691.
10. I. Juhász, K. Kunen and M. E. Rudin, *Two more hereditarily separable non-Lindelöf spaces*, Canad. J. Math. **28** (1976), 998–1005.
11. K. Nagami, *A normal space Z with $\text{ind } Z = 0$, $\text{dim } Z = 1$, $\text{Ind } Z = 2$* , J. Math. Soc. Japan **18** (1966), 158–165.
12. E. Pol and R. Pol, *A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an N -compact space of positive dimension*, Fund. Math. **97** (1977), 43–50.
13. T. Przymusiński, *On the dimension of product spaces and an example of M. Wage*, Proc. Amer. Math. Soc. **76** (1979), 315–321.
14. P. Røys, *Failure of equivalence of dimension concepts for metric spaces*, Bull. Amer. Math. Soc. **68** (1962), 609–613.
15. M. Wage, *The dimension of product spaces*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 4671–4672.
16. R. Engelking, *Dimension theory*, North-Holland, Amsterdam, 1978.
17. A. R. Pears, *Dimension theory of general spaces*, Cambridge University Press, Cambridge, 1975.

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