A NOTE ON THE BOTT VANISHING THEOREM

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ABSTRACT. We give a simple example of a foliated 4-manifold that shows that the bound in Bott's theorem is best possible.

Conventions. The following hold unless stated otherwise:

- (1) All manifolds are closed, oriented, and C^∞ .
- (2) All H^* are with $\mathbf{R}$ -coefficients.

1. Introduction. Let M^n be an n -manifold with a foliation $\mathcal{F}^q$ of codimension q . Let $\nu(\mathcal{F})$ denote the normal bundle of $\mathcal{F}$ and $\text{Pont}^*(\nu(\mathcal{F}))$ the Pontrjagin algebra of $\nu(\mathcal{F})$.

(1.1) **THE BOTT VANISHING THEOREM.** $\text{Pont}^i(\nu(\mathcal{F})) = 0$ for $i > 2q$.

This very powerful theorem gives a necessary condition for a subbundle of TM , the tangent bundle of M , to be the tangent bundle $T\mathcal{F}$ of some foliation $\mathcal{F}$. In searching the literature this author was unable to find any example that showed that (1.1) was best possible in terms of dimension. However, in [T1] Thurston proved the existence of such a foliation.

(1.2) **PROBLEM.** Find $(M^n, \mathcal{F}^q)$ such that $\text{Pont}^{2q}(\nu(\mathcal{F})) \neq 0$.

No claims of uniqueness are made for the example in this paper—just simplicity. A powerful theorem of Thurston is used in constructing the example.

(1.3) **THEOREM (T2, COROLLARY 3).** *Every 2-plane field on M^n , $n > 3$, is homotopic to a completely integrable C^∞ -plane field.*

In other words, we can always view a 2-dimensional subbundle of TM^n , $n > 3$, as the tangent bundle of a foliation (up to bundle isomorphism).

We will construct a 4-manifold M^4 with a 2-frame field, which certainly gives us a trivial 2-plane field. By (1.3) we can take this plane field as $T(\mathcal{F}^2) \cong \epsilon^2$. Since $T(\mathcal{F}^2) \oplus \nu(\mathcal{F}^2) \cong TM^4$ we have that $p_1(\nu(\mathcal{F})) = p_1(M)$. If we can show that $p_1(M) \neq 0$ then we will have shown that $\text{Pont}^4(\nu(\mathcal{F})) \neq 0$, which solves (1.2).

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2. Example. In this section all manifolds are 4-dimensional.

Let $\mathcal{S}$ be the nonsingular bilinear pairing from

$$H^2(M; \mathbf{Z})/\text{torsion} \otimes H^2(M; \mathbf{Z})/\text{torsion} \rightarrow \mathbf{Z}$$

coming from Poincaré duality. We say that $\omega \in H^2(M; \mathbf{Z})/\text{torsion}$ is a *Wu class* if $\mathcal{S}(\omega, x) \equiv \mathcal{S}(x, x) \pmod{2}$, for all $x \in H^2(M; \mathbf{Z})/\text{torsion}$. Let Ω be the set $\{\mathcal{S}(\omega, \omega)\} \subset \mathbf{Z}$.

The following result of Hirzebruch and Hopf is found in [Th, Theorem 19].

(2.1) PROPOSITION. M^4 admits a 2-frame field with finite singularities. The following pairs of integers, and only these, occur as the index for such a 2-frame field:

$$a = \frac{1}{4}(\alpha - 3\sigma - 2\chi), \quad b = \frac{1}{4}(\alpha - 3\sigma + 2\chi),$$

where σ is the signature of M^4 , χ is the Euler characteristic, and $\alpha \in \Omega$.

We will now construct a specific 4-manifold. Let

$$\tilde{M}^4 = \mathbf{C}P^2 \# \mathbf{C}P^2 \# \mathbf{C}P^2 \# \mathbf{C}P^2 \# T^4 \# T^4 \# T^4,$$

where $\#$ denotes the connected sum and T^4 is the 4-torus. Easy calculations show that $\chi(\tilde{M}^4) = 0$, and $\sigma(\tilde{M}^4) = 4$. Hence, our obstruction to a 2-frame field is $\frac{1}{4}(\alpha - 12)$. Noting that $\mathcal{S}$ treats the connected sum operation like disjoint union, simple linear algebra shows that we may choose $\alpha = 12$. Therefore, $\tilde{M}^4$ has a 2-frame field, and we are done since $p_1(\tilde{M}^4) \neq 0$.

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