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PERIODIC POINTS OF POSITIVELY EXPANSIVE MAPS

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ABSTRACT. Stability of fixed points of a uniformly convergent sequence of open ε-locally expansive maps of a compact connected locally connected metric space is studied by Hu and Rosen. Under the same assumption we prove a stability result stronger than theirs.

Hu and Rosen proved [4] a stability theorem for fixed points of a uniformly convergent sequence of open ε-locally expansive maps of a compact connected locally connected metric space. We prove a stronger stability theorem by using the pseudo-orbit tracing property.

Let (X, d) be a compact metric space, and let $f\colon X \to X$ be a continuous map. We assume that f is onto. We say that f is *positively expansive* if there exists a number $c > 0$ such that $x \neq y$ implies the existence of a positive integer n such that $d(f^n(x), f^n(y)) > c$; c is called a *positively expansive constant* for f. We say that f is *ε-locally expansive* if there is an $\varepsilon > 0$ such that $0 < d(x, y) < \varepsilon$ implies $d(f(x), f(y)) > d(x, y)$. We say that f is an *ε-local expansion* with skewness λ if there exist $\varepsilon > 0$ and $\lambda > 1$ such that $0 < d(x, y) < \varepsilon$ implies $d(f(x), f(y)) > \lambda d(x, y)$. In [5] Reddy proved that if f is ε-locally expansive, then f is positively expansive with constant $\varepsilon/2$, and there is a compatible metric ρ for X such that f is an ε_1-local expansion with skewness λ under ρ for some constants $\varepsilon_1 > 0$ and $\lambda > 1$. Rosenholtz showed in [7] that if f is an open ε-local expansion and if X is connected, then f has a fixed point. Given $\delta > 0$, a sequence $\{x_j\}_{j=0}^n$ $(0 \leqslant n \leqslant \infty)$ is called a *δ-pseudo-orbit* (δ-p.o.) of f if $d(f(x_j), x_{j+1}) < \delta$ for $0 \leqslant j \leqslant n-1$. Given $\eta > 0$, a sequence $\{x_j\}_{j=0}^n$ is said to be *η-traced* by a point y in X if $d(f^j(y), x_j) < \eta$ for $0 \leqslant j \leqslant n$. We say that f has the *pseudo-orbit tracing property* (P.O.T.P.) if for each $\eta > 0$ there is $\delta > 0$ such that every δ-p.o. of f can be η-traced by some point in X. We remark that this property is independent of the metric for X. It is well known that if f is an open ε-local expansion, then f has the P.O.T.P. (cf. Proposition 3.6 of [2]).

The following is proved.

THEOREM. *Let (X, d) be a compact connected locally connected metric space, and let $f_i\colon X \to X$ be open ε-locally expansive maps for $i = 0, 1, 2, \ldots$ such that the sequence*

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$\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. Then for any $\eta > 0$ there is a positive integer N such that, for each $n \geqslant 1$ and each fixed point a_0 of f_0^n, there exists a sequence of fixed points a_i of f_i^n with $d(a_i, a_0) < \eta$ for each $i \geqslant N$.

First of all we prepare lemmas that we shall need.

LEMMA 1. *Let (X, ρ) be a compact metric space, and let $f\colon X \to X$ be an open ε_1-local expansion with skewness λ. Then there is a positive number $\delta_0 < \varepsilon_1/2$ such that $x, y \in X$ and $\rho(f(x), y) < \delta_0$ implies $B_{\delta_0/\lambda}(x) \cap f^{-1}(y) \neq \varnothing$, where $B_\alpha(x) = \{z \in X\colon \rho(x, z) < \alpha\}$.*

PROOF. See Lemma 1 of [3].

LEMMA 2. *Let (X, ρ) be a compact connected metric space, and let $f\colon X \to X$ be an open ε_1-local expansion with skewness λ. Then* per(f), *the set of all periodic points of f, is dense in X.*

PROOF. Let δ_0 be as in Lemma 1. It is enough to see that for each $\nu > 0$ ($\nu < \delta_0/2$) there is a positive integer r such that f^r has a fixed point in $B_\nu(x)$. For each number $\nu > 0$, choose $\delta > 0$ as in the definition of the P.O.T.P. Let $\mathscr{U} = \{U_1, U_2, \dots, U_n\}$ be a finite open cover of X so that the diameter of each $U_i \in \mathscr{U}$ is less than δ_0. Then we can choose a positive integer r such that $n\delta_0/\lambda^{r-1} < \delta$. Take and fix $x_0 \in X$. Since X is connected, there is a finite sequence $\{y_0 = x_0, y_1, \dots, y_k, y_{k+1} = f^r(x_0)\}$ such that $y_j, y_{j+1} \in U_{i_j} \in \mathscr{U}$ for $0 \leqslant j \leqslant k$ ($\leqslant n$). By Lemma 1 there is y_k^1 in X with the property that $f(y_k^1) = y_k$ and $\rho(y_k^1, f^{r-1}(x_0)) < \delta_0/\lambda$ (since $\rho(y_k, f(f^{r-1}(x_0))) < \delta_0$). Similarly, since $\rho(y_{k-1}, f(y_k^1)) < \delta_0$, there is y_{k-1}^1 with the property that $f(y_{k-1}^1) = y_{k-1}$ and $\rho(y_{k-1}^1, y_k^1) < \delta_0/\lambda$. Continuing in this fashion, we can construct a finite sequence $\{y_0^1 = x_1, y_1^1, \dots, y_k^1, y_{k+1}^1 = f^{r-1}(x_0)\}$. Next, since $\rho(y_k^1, f^{r-1}(x_0)) < \delta_0$, by Lemma 1 there is y_k^2 in X with the property that $f(y_k^2) = y_k^1$ and $\rho(y_k^2, f^{r-2}(x_0)) < \delta_0/\lambda^2$. Similarly, since $\rho(y_{k-1}^1, f(y_k^2)) < \delta_0/\lambda$, there is y_{k-1}^2 in X with the property that $f(y_{k-1}^2) = y_{k-1}^1$ and $\rho(y_{k-1}^2, y_k^2) < \delta_0/\lambda^2$. In this manner we can find a finite sequence $\{y_0^2 = x_2, y_1^2, \dots, y_k^2, y_{k-1}^2 = f^{r-2}(x_0)\}$. Inductively, for $1 \leqslant l \leqslant r-1$, we can construct a finite sequence $\{y_0^l = x_l, y_1^l, \dots, y_k^l, y_{k+1}^l = f^{r-l}(x_0)\}$. Then

$$\rho(x_{r-1}, f(x_0)) \leqslant \rho\big(y_0^{r-1}, y_1^{r-1}\big) + \cdots + \rho\big(y_k^{r-1}, f(x_0)\big) \leqslant n\delta_0/\lambda^{r-1} < \delta.$$

Hence, $\{x_0, x_{r-1}, x_{r-2}, \dots, x_1, x_0, x_{r-1}, \dots\}$ is a δ-p.o. of f. Since f has the P.O.T.P. and f is positively expansive with constant δ_0, there is a tracing point p in $B_\nu(x_0)$ such that $f^r(p) = p$ (cf. [1]).

LEMMA 3. *Let (X, d) be a compact connected metric space, and let $f\colon X \to X$ be open and ε-locally expansive. If E is a closed nonempty subset of X with $f^{-1}(E) \subset E$, then $E = X$.*

PROOF. There exist a compatible metric ρ for X and numbers $\varepsilon_1 > 0$ and $\lambda > 1$ such that f is an open ε_1-local expansion with skewness λ under ρ. To arrive at the conclusion, it is enough to prove that per$(f) \subset E$. Let $p \in$ per(f) with $f^r(p) = p$. Since X is compact, there are $\varepsilon' > 0$ and $\lambda' > 1$ such that f^r is an open ε'-local expansion with skewness λ' under ρ. Hence, we can find $\delta_0' > 0$ satisfying Lemma 1 for f^r. Let $\mathcal{U}' = \{U_1', U_2', \ldots, U_{n'}'\}$ be a finite open cover of X so that the diameter of each $U_{i_j}' \in \mathcal{U}'$ is less than δ_0' under ρ. Take $y_0 \in E$. Since X is connected, there is a finite sequence $\{z_0 = y_0, z_1, \ldots, z_{k'}, z_{k'+1} = f^r(p) = p\}$ such that $z_j, z_{j+1} \in U_{i_j}'$ for $0 \leqslant j \leqslant k'$ $(\leqslant n')$. By the argument used in the proof of Lemma 2, for $m > 0$ we can construct a finite sequence $\{z_0^m = y_m, z_1^m, \ldots, z_{k'}^m, z_{k'+1}^m = f^r(p) = p\}$ with the property that $y_m \in f^{-rm}(y_0) \subset E$ and $\rho(y_m, z_1^m) + \cdots + \rho(z_{k'}^m, p) \leqslant n'\delta_0'/\lambda'^m$. Hence, $y_m \to p$ $(m \to \infty)$, so $\mathrm{per}(f) \subset E$.

The following lemma is essentially due to Hu and Rosen [4].

Lemma 4. *Let (X, d) be a compact connected locally connected metric space, and let $f_i\colon X \to X$ be open ε-locally expansive maps for $i = 0, 1, 2, \ldots$ such that the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. Then there is an integer $N_1 > 0$ such that $i \geqslant N_1$ implies* $\mathrm{card}(f_i^{-1}(x)) = \mathrm{card}(f_0^{-1}(y))$ *for all $x, y \in X$.*

Proof. According to Lemma 2 in [6], there is a finite open cover $\{W_\beta\}$ of X such that, for each β and for $i = 0, 1, 2, \ldots$, $\operatorname{diam} W_\beta < \varepsilon/2$ and f_i maps every component of $f_i^{-1}(W_\beta)$ homeomorphically onto W_β. Let α be a Lebesgue number for $\{W_\beta\}$. It is easy to check that if $x, y \in X$ and $d(f_i(x), y) < \alpha$, then $B_\alpha'(x) \cap f_i^{-1}(y)$ consists of a single point for each i by the ε-local expansiveness of f_i. Here $B_\alpha'(x) = \{z \in X\colon d(x, z) < \alpha\}$. Since $f_i \to f_0$ uniformly, there is an N_1 such that $i \geqslant N_1$ implies $d(f_i(x), f_0(x)) < \alpha$ for all $x \in X$. Fix $x_0 \in X$. Now suppose $i \geqslant N_1$ and $z \in f_0^{-1}(x_0)$. Then $d(f_i(z), x_0) < \alpha$. Thus, there is a unique $x' \in X$ per z such that $d(z, x') < \alpha < \varepsilon/2$ and $x' \in f_i^{-1}(x_0)$. Hence, $\mathrm{card}(f_i^{-1}(x_0)) \geqslant \mathrm{card}(f_0^{-1}(x_0))$. A similar argument shows that $\mathrm{card}(f_i^{-1}(x_0)) \leqslant \mathrm{card}(f_0^{-1}(x_0))$. By Property 4.5 in [4], we get the conclusion.

Proof of Theorem. Take η with $0 < \eta < \varepsilon/6$. Let $\delta > 0$ be as in the definition of the P.O.T.P. of f_0. Since $f_i \to f_0$ uniformly, there is an integer $N \geqslant N_1$ such that $i \geqslant N$ implies $d(f_i(x), f_0(x)) < \delta$ for all $x \in X$. Here N_1 is an integer given by Lemma 4. Then for $i \geqslant N$ there are unique continuous one-to-one (not necessarily onto) maps $h_i\colon X \to X$ with $h_i f_i = f_0 h_i$ and $d(h_i(x), x) < \eta$ for all $x \in X$ (cf. [8, pp. 236–238]). By Lemma 4 we easily obtain that $f_0^{-1}h_i(X) \subset h_i(X)$ for $i \geqslant N$. Indeed, we may assume that there is an integer k such that, for all $i \geqslant N$ and for all $x \in X$, $\mathrm{card}(f_i^{-1}(x)) = \mathrm{card}(f_0^{-1}(x)) = k$. Hence, for each $x' \in h_i(X)$, there are exactly k-points $\{y_1, y_2, \ldots, y_k\}$ in X such that $f_i(y_m) = h_i^{-1}(x')$ for $1 \leqslant m \leqslant k$ (since h_i is one-to-one). Obviously, $f_0(h_i(y_m)) = h_i(f_i(y_m)) = x'$ for $1 \leqslant m \leqslant k$. That is,

$$f_0^{-1}(x') = \{h_i(y_1), h_i(y_2), \ldots, h_i(y_k)\} \subset h_i(X)$$

(since h_i is one-to-one and $\mathrm{card}(f_0^{-1}(x')) = k$). Hence, by Lemma 3 we have $h_i(X) = X$ for $i \geqslant N$. Let a_0 be a fixed point of f_0^n and put $a_i = h_i^{-1}(a_0)$. Then $f_i^n h_i^{-1} = h_i^{-1} f_0^n$ implies $f_i^n(a_i) = a_i$, and $d(h_i(x), x) < \eta$ for all $x \in X$ implies $d(a_i, a_0) < \eta$ for all $i \geqslant N$ and for all $n \geqslant 1$. $\square$

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