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INTRINSIC CURVATURE OF THE INDUCED METRIC ON HARMONICALLY IMMERSED SURFACES[1]

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ABSTRACT. A theorem by Wissler is used to prove the following result. Suppose that an oriented surface S with indefinite prescribed metric h is harmonically mapped into an arbitrary pseudo-Riemannian manifold so that the metric I induced on S is complete and Riemannian. Then the intrinsic curvature $K(I)$ of the immersion satisfies $\inf|K(I)| = 0$, with $\sup|\operatorname{grad} 1/K(I) = \infty$ in case $K(I)$ never vanishes on S.

1. Suppose that an oriented surface S is provided with an indefinite prescribed metric h. Suppose further that S is harmonically immersed in some pseudo-Riemannian manifold M so that the metric I induced on S is complete and Riemannian. We show in this paper that the intrinsic curvature $K(I)$ cannot be bounded away from zero on S.

Note that the pseudo-Riemannian manifold M in which S is immersed is quite arbitrary, and in particular, may be Riemannian. Despite the generality of the ambient manifold, we have a result strikingly similar to Efimov's theorem that $K(I)$ cannot be bounded away from zero on a complete, noncompact surface in Euclidean 3-space E^3.

Throughout the study of harmonically immersed surfaces, there are echos of classical surface theory. When the prescribed metric h is indefinite, there is a tie to the line of ideas which sprang from the study of surfaces in E^3 with constant negative Gauss curvature. For some insight into this phenomenon, see **[10]**.

Because an indefinite metric h is prescribed on the surface S above, we know that S cannot be compact with genus zero. Thus a result of Bonnet-Hopf-Rinow (see **[13]**) applied to S with the complete Riemannian metric I implies that $K(I)$ must be negative if bounded away from zero. To show that a negative $K(I)$ cannot be bounded away from zero, one must use the hypothesis that S is harmonically immersed.

The proof involves straightforward application of a theorem due to Wissler **[14]**. In all but the most degenerate cases, an h-null Tchebychev net (defined below) automatically exists on a harmonically immersed surface with indefinite prescribed metric h. Wissler's theorem states that an abstract surface provided with a complete

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Riemannian metric and a Tchebychev net cannot have intrinsic curvature bounded away from zero.

To get his result, Wissler generalized arguments used by Hilbert **[6]** and Holmgren **[7]** at the turn of the century to show that the hyperbolic plane cannot be isometrically immersed in E^3. Wissler's theorem may have seemed a somewhat artificial extension of known results established by pushing classical arguments as far as they would go in directions suggested by Efimov's work **[2–4]**. Yet application of Wissler's theorem to harmonically immersed surfaces with indefinite prescribed and Riemannian induced metric is completely natural. Indeed, the real work needed to prove the theorem below has been done in **[10]** and **[14]**.

In §2 we present a more detailed statement of Wissler's theorem, of our result, and of the argument just outlined. Though less is needed, we assume C^∞ smoothness throughout. For the definition of a harmonic map, see **[10]**.

2. A Tchebychev net is formed on an orientable surface S with Riemannian metric g by the null curves of an indefinite real quadratic form $b = b_{ij}dx^i dx^j$, $i, j = 1, 2$, in case there are local coordinates x, y available anywhere on S in terms of which

$$g = dx^2 + 2\cos\omega \, dx\, dy + dy^2, \quad b = 2b_{12}\, dx\, dy.$$

Such coordinates x, y are called Tchebychev coordinates. Opposite sides of Tchebychev coordinate curve quadralaterals have equal lengths. The function ω can be taken so that $0 < \omega < \pi$, and represents the angle between the h-null Tchebychev coordinate curves.

The existence of Tchebychev nets on any harmonically immersed surface with indefinite prescribed metric was suggested by an example well known to physicists. There (see **[5]** or **[12]**) the Minkowski 2-plane is harmonically immersed in the standard round 2-sphere. The general situation is described by Theorem 7 in **[10]**. When the induced metric I is Riemannian, we have the following.

LEMMA 1. *Given a harmonic map from an oriented surface S with indefinite prescribed metric h and Riemannian induced metric I, the h-null curves form a Tchebychev net on S with respect to I.*

PROOF. Note that the map must be an immersion since the induced metric I is nondegenerate. Given our hypotheses, Theorem B from **[10]** states that the equiareal metric

$$\Pi = \sqrt{-\det I/\det h}\, h$$

satisfies the Codazzi Mainardi equations of classical surface theory with respect to the Riemannian metric I. Thus Theorem 2 from **[14]** assures that the curves defined by setting $h = 0$ provide a Tchebychev net on S with respect to I. We should have cited Wissler's Theorem 2 from **[14]** in connection with Lemma 10 from **[9]** and Lemma 8 from **[10]**. However, we were unaware of his result at the time.

REMARK. Using the coordinates x, y provided by Lemma 1, the pair of forms

$$I = dx^2 + 2\cos\omega\, dx\, dy + dy^2, \qquad II = \pm 2\sin\omega\, dx\, dy$$

looks much like the pair of fundamental forms I, II for a surface in E^3 with Gauss curvature $\mathscr{K} \equiv -1$. This is not accidental. For one thing, Π is a conformal normalization of h, chosen so that $(\det \Pi/\det I) \equiv -1$. However, the crucial fact (see **[10]**) is that the identity map from S with prescribed metric II to S with prescribed metric I is harmonic for an S in E^3 with $\mathscr{K}$ a negative constant.

It was the global behavior of the Tchebychev net that caused some trouble in Hilbert's original proof **[6]**. We will adapt our argument from Appendix 1 of **[8]** to show that here the Tchebychev net lifted to the universal cover $\tilde{S}$ of S must be globally cartesian if I is complete. Alternately, one might use Theorem 3 from **[14]**. But we do not know from the outset here that opposite sides of a Tchebychev net quadralateral must be equal in length, unless the quadralateral is in the domain of some local Tchebychev coordinates.

LEMMA 2. *Suppose the induced metric I is complete in Lemma* 1. *Then the lift of the Tchebychev net defined by $h = 0$ to the universal cover $\tilde{S}$ of S is globally cartesian, i.e., there is a diffeomorphism of $\tilde{S}$ on to E^2 taking the Tchebychev net to the cartesian coordinate net.*

PROOF. Consider the equivalent Γ of the third fundamental form given by

$$\Gamma = (\mathrm{tr}_I \Pi)\Pi + I.$$

The form Γ is positive definite because $0 < \omega < \pi$ and

$$\Gamma = dx^2 - 2\cos\omega \, dx\, dy + dy^2$$

in terms of Tchebychev coordinates. Thus the form

$$\Lambda = \tfrac{1}{2}(I + \Gamma) = dx^2 + dy^2$$

is complete on $\tilde{S}$, since $2\Lambda > I$. Note that the Tchebychev net is an orthogonal net of geodesics for the metric Λ on $\tilde{S}$. There must be an orientation preserving isometry from the simply connected, flat Riemannian manifold $(\tilde{S}, \Lambda)$ onto E^2. (See p. 149 of **[1]**.) Under this isometry, the Tchebychev net goes to a net of mutually perpendicular straight lines. Translating and rotating E^2 if necessary, one gets the result claimed.

Lemma 2 guarantees that when I is complete in Lemma 1, there is a global set of Tchebychev coordinates on $\tilde{S}$. This implies that opposite sides of all Tchebychev net quadralaterals on S have equal lengths.

We now paraphrase Theorem 7 from **[14]** and state the result which our remarks have established.

WISSLER'S THEOREM. *If a Tchebychev net is defined on an orientable surface with complete Riemannian metric g, then the intrinsic curvature K of g satisfies* $\inf|K| = 0$, *with* $\sup|\mathrm{grad}(1/K)| = \infty$ *in case K never vanishes.*

THEOREM. *Suppose a harmonic map from an oriented surface S with indefinite prescribed metric h into an arbitrary pseudo-Riemannian manifold has a complete Riemannian induced metric I. Then the intrinsic curvature $K(I)$ of the immersion satisfies* $\inf|K(I)| = 0$, *with* $\sup|\mathrm{grad}\{1/K(I)\}| = \infty$ *in case $K(I) \neq 0$ on S.*

By the Remark above, Hilbert's theorem [**6**] is a special case of our Theorem. At this point, we can suggest only the most obvious applications of the Theorem, such as those which can be read directly from the Gauss curvature equation. (See Corollary 4.6 in [**11**].) Finally, for the harmonic immersion of the Minkowski 2-plane in the round 2-sphere cited in [**5**] and [**12**], our Theorem states that I cannot be complete, since $K(I) \equiv 1$ there.

REFERENCES

1. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

2. N. V. Efimov, *Appearance of singularities on surfaces of negative curvature*, Mat. Sb. **64** (1964), 286–320; English Transl., Amer. Math. Soc. Transl. **66** (1968), 154–190.

3. ______, *Differential criteria for homeomorphism of certain mappings with applications to the theory of surfaces*, Mat. Sb. **76** (1968), 499–512; English Transl., Math USSR-Sb. **5** (1968), 475–488.

4. ______, *Surfaces with slowly changing negative curvature*, Uspekhi Mat. Nauk **21** (1966), 3–58; English Transl. Russian Math. Surveys **21** (1966), 1–56.

5. C. H. Gu, *On the Cauchy problem for harmonic maps defined on two dimensional Minkowski space*, Comm. Pure Appl. Math. **33** (1980), 727–738.

6. D. Hilbert, *Grundlagen der Geometrie*, vol. 9, Stuttgart, 1962.

7. E. Holmgren, *Sur les surfaces a courbre constante negative*, C.R. Acad. Sci. Paris **13 4** (1902), 740–743.

8. T. K. Milnor, *Efimov's theorem about complete immersed surfaces of negative curvature*, Adv. in Math. **8** (1972), 474–543.

9. ______, *Abstract Weingarten surfaces*, J. Differential Geometry **15** (1980), 365–380.

10. ______, *Harmonic maps and classical surface theory in Minkowski 3-space*, Trans. Amer. Math. Soc. **280** (1983), 161–185.

11. B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, 1983.

12. K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through quadratic constraints*, Comm. Math. Phys. **46** (1976), 207–221.

13. J. J. Stoker, *Differential geometry*, Wiley-Interscience, New York, 1969.

14. C. Wissler, *Globale Tschebyscheff-Netze auf Riemannischen Mannigfaltkeiten und Fortsetzung von Flächen konstanter negativer Krummung*, Comm. Math. Helv. **47** (1972), 348–372.

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