THE CATENARIAN PROPERTY OF POWER SERIES RINGS OVER A PRÜFER DOMAIN

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ABSTRACT. Let D be a Prüfer domain that has the SFT-property. It is shown that the power series ring $D[[x]]$ is catenarian. If $n > 1$ and $\dim D > 1$ then the ring $D[[x_1, \dots, x_n]]$ is not catenarian.

1. Introduction. Recall that a ring R is a *catenary ring* if for each saturated chain $P_0 < P_1 < \dots < P_n$ of prime ideals of R we have $\text{rank}(P_n/P_0) = n$. In particular, an integral domain D is catenarian if, for every prime ideal P of D , the length of every saturated chain of prime ideals between (0) and P equals $\text{rank } P$. Based on the work of Ratliff in [9] and Seydi in [10], Lequain has shown in [7] that if D is a Noetherian integral domain then the domains $D[x]$, $D[x_1, \dots, x_n]$, $D[[x]]$, and $D[[x_1, \dots, x_n]]$ are simultaneously catenarian.

In [4] and [5] it is shown that if D is a finite-dimensional Bezout domain then $D[x]$ is catenarian. More recently, De Souza Doering and Lequain have shown in [8] that if D is a finite dimensional Prüfer domain then $D[x]$ is catenarian. Finally, Bouvier and Fontana have shown in [6] that if D is a locally finite-dimensional Prüfer domain then $D[x_1, \dots, x_n]$ is catenarian for each $n \geq 1$. In view of the above referenced results for Noetherian integral domains, it is natural to inquire about the catenarian properties of $D[[x]]$ and $D[[x_1, \dots, x_n]]$ when D is a Prüfer domain.

In [1] the author has defined an ideal A of a ring R to be of *strong finite type* (or an SFT-ideal) provided there is a finitely generated ideal $B \subset A$ and a positive integer k such that $a^k \in B$ for each $a \in A$. The ring R satisfies the SFT-property provided each ideal of R is an SFT-ideal. It is shown in [1] (cf. the proof of [1, Theorem 1]) that if a ring R does not have the SFT-property then there exists an infinite chain of prime ideals in $R[[X]]$. We must, therefore, restrict our attention to Prüfer domains with the SFT-property. Such domains have already been studied in [2 and 3].

2. The catenarian property in $D[[x]]$. Throughout this section, D denotes a Prüfer domain with the SFT-property and K denotes the quotient field of D . We let $X_n = \{x_1, \dots, x_n\}$ and we write $D[[X_n]]$ to denote $D[[x_1, \dots, x_n]]$.

LEMMA 1. *Let Q_1 and Q_2 be prime ideals of $D[[X_n]]$ such that $Q_2 < Q_1$, $\text{rank}(Q_1/Q_2) = 1$, and $Q_i \cap D = P_i$, $i = 1, 2$. Then either $P_1 = P_2$ or $\text{rank}(P_1/P_2) = 1$ and $Q_1 \subset P_1 + (X_n)$.*

PROOF. Suppose that $P_2 < P_1$, let $p \in P_1 \setminus P_1^2$ (cf. [2, Lemma 2.7]), and let W be a valuation overring of $D[[X_n]]$ with prime ideals $N_2 < N_1$ such that $N_i \cap D[[X_n]] = Q_i$, $i = 1, 2$, N_1 is maximal in W , and $N_1 = \text{rad}(pW)$. Since $V = W \cap K$ is a

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valuation overring of D centered on P_1 we must have $V = D_{P_1}$. Assume there exists a prime ideal P of D such that $P_2 < P < P_1$. Then $PV \subset \bigcap_{k=1}^{\infty} p^k V$ so $PW \subset \bigcap_{k=1}^{\infty} p^k W$. If we set $N = \bigcap_{k=1}^{\infty} p^k W$ then N is a prime ideal of W and, since $P \subset N \cap D < P_1$, it follows that $Q_2 < N \cap D[[X_n]] < Q_1$. But this contradicts our assumption that $\text{rank}(Q_1/Q_2) = 1$, so we conclude that $\text{rank}(P_1/P_2) = 1$.

Let $\mathcal{S}$ denote the set of invertible ideals A of D such that $P_1 < A$. As in [3, p. 901], we see that $\mathcal{S}$ is a multiplicatively closed set of invertible ideals and $D_{\mathcal{S}} = \{\xi \in K \mid \xi A \subset D \text{ for some } A \in \mathcal{S}\}$ is a flat overring of D with prime ideals $\{PD_{\mathcal{S}} \mid P \text{ is a prime ideal of } D \text{ and } P \not\prec P_1\}$. In particular, $P_1 D_{\mathcal{S}}$ is a maximal ideal of $D_{\mathcal{S}}$ and, since $P_1 D_{\mathcal{S}} \cap D = P_1$ and $P_1 D_{\mathcal{S}} \subset D$, we have $P_1 = P_1 D_{\mathcal{S}}$. Following the proof of [3, Theorem 3.6] we see that $W \supset D_{\mathcal{S}}[[X_n]]$. Set $\mathcal{P}_1 = N_1 \cap D_{\mathcal{S}}[[X_n]]$. Then $\mathcal{P}_1 \cap D_{\mathcal{S}} = P_1 D_{\mathcal{S}}$, so $\mathcal{P}_1 \subset [P_1 + (X_n)]D_{\mathcal{S}}[[X_n]]$. Therefore, $Q_1 = \mathcal{P}_1 \cap D[[X_n]] \subset P_1 + (X_n)$.

For a prime ideal Q of $D[[x]]$, let $\mathcal{A}(Q)$ denote the set of prime ideals Q' of $D[[x]]$ such that $Q < Q'$ and $\text{rank}(Q'/Q) = 1$. Similarly, $\mathcal{B}(Q)$ denotes the set of prime ideals Q' of $D[[x]]$ such that $Q' < Q$ and $\text{rank}(Q/Q') = 1$. Finally, $I(Q)$ denotes the ideal of D consisting of the constant terms of elements of Q .

LEMMA 2. *Let Q be a prime ideal of $D[[x]]$ such that $Q \cap D = P$, $P[[x]] < Q$, and $Q \neq P + (x)$. Then*

- (1) $\mathcal{A}(Q)$ is a finite set and $\mathcal{A}(Q) = \{P' + (x) \mid P' \text{ is minimal over } I(Q)\} \subset \{P' + (x) \mid \text{rank}(P'/P) = 1\}$ and
- (2) $P[[x]]$ is the unique element in $\mathcal{B}(Q)$.

PROOF. Let Q' be a prime ideal of $D[[x]]$ and set $P' = Q' \cap D$. Then $P'[[x]] \subset Q'$ by [1, Theorem 1] and, by applying Lemma 3.5 of [3] to $(D/P')[[x]]$, we see that either $Q' = P'[[x]]$ or $\text{rank}(Q'/P'[[x]]) = 1$. If $Q' \in \mathcal{A}(Q)$ it follows that $P < P'$, so, by Lemma 1, $\text{rank}(P'/P) = 1$ and $Q' \subset P' + (x)$. But $Q' \neq P'[[x]]$ by [2, Corollary 3.6] so $Q' = P' + (x)$. Since $Q < Q'$ it is necessary that $I(Q) \subset P'$. But $I(Q) \not\subset P$ (otherwise we would have $Q \subset P + (x)$), so P' is minimal over $I(Q)$.

Conversely, let P' be a prime ideal of D such that P' is minimal over $I(Q)$. Then $Q < P' + (x)$, so there exists a prime ideal Q'' of $D[[x]]$ such that $Q < Q'' \subset P' + (x)$ and $Q'' \in \mathcal{A}(Q)$. As we have just shown, $Q'' = P'' + (x)$ for some prime ideal P'' of D such that $\text{rank}(P''/P) = 1$ and P'' is minimal over $I(Q)$. It follows that $P' = P''$ and, hence, that $P' + (x) \in \mathcal{A}(Q)$. The finiteness of $\mathcal{A}(Q)$ follows from Corollary 2.6 of [2].

Now suppose that $Q' \in \mathcal{B}(Q)$ and $P' = Q' \cap D$. Since $Q \not\subset P + (x)$, it follows from Lemma 1 that $P = P'$. Consequently, $Q' = P[[x]]$.

LEMMA 3. *Let P be a nonzero prime ideal of D and let $(0) < Q_1 < \cdots < Q_s < P + (x)$ be a saturated chain of prime ideals in $D[[x]]$. Then $s = \text{rank } P$.*

PROOF. Suppose that $s > 1$ and let P_1 be the unique rank one prime ideal of D that is contained in P . Either $Q_1 = P_1[[x]]$ or $Q_1 \cap D = (0)$. Since $Q_2 \cap D \neq (0)$ by [3, Lemma 3.5], $Q_1 \cap D = (0)$ implies, by Lemma 1, that $Q_2 \cap D = P_1$. Thus, since $\text{rank } P_1[[x]] = 1$, in either case we have that $Q_2 > P_1[[x]]$. Since $P + (x) > Q_2$ by assumption, it follows that $P \neq P_1$; that is, $\text{rank } P \geq 2$. Thus, if $\text{rank } P = 1$, then $s = 1$. On the other hand, if $\text{rank } P > 1$ we may pass to D/P_1 and argue, by induction, that $s = \text{rank } P$.

THEOREM 4. *If D is a Prüfer domain with the SFT-property, then $D[[x]]$ is catenarian.*

PROOF. Let Q be a nonzero prime ideal of $D[[x]]$ with $P = Q \cap D$. If $Q = P[[x]]$ then, by [2, Corollary 3.6], there is a unique chain of prime ideals between (0) and Q . Likewise, if $Q > P[[x]]$ but $Q \neq P + (x)$, then, by Lemma 2(2) and [2, Corollary 3.6], there is a unique chain of prime ideals between (0) and Q . The remaining case, $Q = P + (x)$, has already been handled in Lemma 3.

If $\dim D = 1$ then D is a Dedekind domain (cf. [2, p. 4]) and hence, by [7, Theorem 2.6], $D[[X_n]]$ is catenarian for each n . If $\dim D = m$ then, by [3, Theorem 3.6], $\dim D[[X_n]] = mn + 1$. Indeed, if M is a maximal ideal of D and $\text{rank } M = m$, then $\text{rank}(M + (X_n)) = mn + 1$. But $\text{rank}(x_n) = 1$ and

$$\text{rank}[(M + (X_n))/(x_n)] = \text{rank}(M + (X_{n-1})) = m(n-1) + 1,$$

so if $m > 1$ then each saturated chain of the form $(0) < (x_n) < \cdots < M + (X_n)$ has length at most $m(n-1) + 2 < mn + 1$. Thus, $D[[X_n]]$ is not catenarian.

3. The valuation ring case. A valuation ring V has the SFT-property if and only if it is a discrete valuation ring. (A valuation ring V is discrete provided $P \neq P^2$ for each prime ideal P of V . Cf. Lemma 2.7 and Proposition 3.1 of [2].) The results of §2 allow us to give a fairly complete description of the prime spectrum of $V[[x]]$ when V is a discrete valuation ring. First, we restate and strengthen Lemma 2 as follows.

PROPOSITION 5. *Let V be a discrete valuation ring and let Q be a prime ideal of $V[[x]]$. If $Q \cap V = P$, $P[[x]] < Q$, and $Q \neq P + (x)$, then*

- (1) $\mathcal{A}(Q) = \{P' + (x)\}$ where P' is the unique prime ideal of V such that $P < P'$ and $\text{rank}(P'/P) = 1$,
- (2) $\mathcal{B}(Q) = \{P[[x]]\}$,
- (3) Q is principal.

PROOF. (1) and (2) are immediate from Lemma 2. We first prove (3) for the case $P = (0)$. In this case $\text{rank } P' = 1$, and if we set $W = V_{P'}$ then $W[[x]]_{W \setminus (0)} = V[[x]]_{V \setminus (0)}$. It follows that $Q' = QV[[x]]_{V \setminus (0)} \cap W[[x]]$ is a prime ideal of $W[[x]]$ such that $Q' \cap W = (0)$ and $Q' \cap V[[x]] = Q$. But $W[[x]]$ is a unique factorization domain so Q' is principal.

Suppose that $Q' = f'W[[x]]$, where $f' = \sum_{i=0}^{\infty} w_i x^i$. Since $Q' \not\subset (P'W)[[x]]$, some w_i , $i \geq 1$, is a unit in W . Thus, we may write $f' = f'_1 + x^k u$ where $k \geq 1$, $f'_1 \in (P'W)[[x]]$, and u is a unit in $W[[x]]$. If $f = f' u^{-1} = f'_1 u^{-1} + x^k = f_1 + x^k$ then $f_1 = f'_1 u^{-1}$ is in $(P'W)[[x]] = P'[[x]] \subset V[[x]]$. Therefore, $f \in V[[x]]$ and $Q' = fW[[x]]$. Let $q \in Q$ and write $q = fh$ where $h \in W[[x]]$. Then $q = f_1 h + x^k h$ and $f_1 h \in P'[[x]] \subset V[[x]]$ so $x^k h = q - f_1 h \in V[[x]]$. It follows that $h \in V[[x]]$ and $Q = fV[[x]]$.

In the general case we may now conclude that $Q = P[[x]] + fV[[x]]$ where, if $f = \sum_{i=0}^{\infty} a_i x^i$, then $a_0 \notin P$. Thus, if $p \in P$ then $p/a_0^n \in V$ for each positive integer n . It follows that $Pf^{-1} \subset V[[x]]$ and, hence, that $P[[x]] \subset fV[[x]]$. Therefore, $Q = fV[[x]]$.

QUESTION. *What is a characterization of the prime elements of $V[[x]]$?*

We conclude with a diagram which illustrates the prime spectrum of a three-dimensional discrete valuation ring V . Let $(0) = P_0 < P_1 < P_2 < P_3$ be the prime ideals of V and let f_i , $0 \leq i \leq 2$, denote arbitrary prime elements of $V[[x]]$ such that $f_i V[[x]] \cap V = P_i$. We use $A \rightarrow B$ to mean $A > B$. We thus have the following diagram of prime ideals in $V[[x]]$.

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