FINE EMBEDDINGS OF FINITE DIMENSIONAL SUBSPACES OF L_p , $1 \leq p < 2$, INTO l_1^m

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ABSTRACT. Every m -dimensional subspace of L_p , $1 < p < 2$, $(1 + \varepsilon)$ -embeds into l_1^n as long as $n \geq \eta m^{1+(1/p)} (\log m)^{-1}$, where $\eta = \eta(p, \varepsilon) < \infty$. For subspaces of L_1 we get a somewhat weaker result.

In [1 and 8] good estimates were obtained for the dimension of the largest l_p^m , $1 \leq p < 2$, space which embeds nicely into l_1^n (in [1]) or into more general finite-dimensional spaces (in [8]).

In this note we are interested in obtaining a similar lower bound for the largest m such that every m -dimensional subspace of L_p , $1 \leq p < 2$, embeds nicely into l_1^n (or into an arbitrary fixed Banach space).

We obtain the following two theorems. $ST_p(U)$ —the stable type p constant of U —is defined below.

THEOREM 1. *For any $1 < p < 2$ and $\alpha > 0$ there exists an $\eta = \eta(p, \alpha) < \infty$ such that if U is a Banach space and m a positive integer satisfying*

$$(ST_p(U))^q \geq \eta \cdot m^{1+1/p} (\log m)^{-1} \quad (1/p + 1/q = 1)$$

then any m -dimensional subspace X of $L_p(0, 1)$ $(1 + \alpha)$ -embeds into U (i.e. there exists a 1-1 operator T from X onto a subspace of U with $\|T\| \|T^{-1}\| \leq 1 + \alpha$).

Since $ST_p(l_1^n)^q = n$ we get that if $n \geq \eta \cdot m^{1+1/p} (\log m)^{-1}$ then every m -dimensional subspace of $L_p(0, 1)$ embeds into l_1^n . For subspaces of $L_1(0, 1)$ we get a worse result.

THEOREM 2. *For each $\alpha > 0$ there exists a $K = K(\alpha)$ such that if $n \geq m^K$ then every m -dimensional subspace of $L_1(0, 1)$ $(1 + \alpha)$ -embeds into l_1^n .*

The proof of the two theorems is based on the proof of the main theorem in [8].

We begin by reviewing some of the ideas and results of [8].

Let $(A_j)_{j \geq 1}$ be an independent and identically distributed (i.i.d.) sequence of exponential random variables, i.e., $P(A_j > t) = e^{-t}$, $t \geq 0$, and let $\Gamma_j = \sum_{n=1}^j A_n$. Let $(y_j)_{j \geq 1}$ be any sequence of symmetric i.i.d. random variable in $L_p(\mu)$, μ a probability measure, $1 \leq p < 2$. Then $X = \sum_{j=1}^\infty (\Gamma_j)^{-1/p} \cdot y_j$ is a symmetric p -stable random

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variable with parameter depending on p and $\|y_1\|_p$ only. In particular, for $p > 1$, $\|X\|_1 = C_p \cdot \|y_1\|_p$ with C_p depending only on p (see [5 and 4]).

Let $1 < p < 2$. The *stable type p constant* of a Banach space U is the smallest C such that

$$E \left\| \sum_{i=1}^n \theta_i x_i \right\| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

for all $x_1, \dots, x_n$ in U , where $\theta_1, \dots, \theta_n$ is a sequence of independent, symmetric, p -stable random variables normalized in L_1 . We denote this constant by $ST_p(U)$. The constant $ST_p(U)$ is also equal to the smallest constant C such that

$$E \left\| \sum_{i=1}^n \theta_i x_i \right\| \leq C \cdot n^{1/p} \cdot \max \{ \|x_i\|, 1 \leq i \leq n \}$$

(see [8]).

Given a Banach space U , $1 < p < 2$, and a set $\{y_a\}_{a \in A}$ of norm one vectors in $L_p(\mu)$, pick a sequence $(x_i)_{i=1}^n$ of vectors in U with $\|x_i\| \leq 1$, $1 \leq i \leq n$, and

$$\frac{1}{n^{1/p}} E \left\| \sum_{i=1}^n \theta_i x_i \right\| \geq \frac{1}{2} ST_p(U).$$

By the contraction principle one can assume $\|x_i\| = 1$ for all $1 \leq i \leq n$ (this is needed in the proof of Theorem 2 only). Define a process $\{Y_a\}_{a \in A}$ taking values in U by

$$P(Y_a = y_a x_i) = P(Y_a = -y_a x_i) = 1/2n, \quad 1 \leq i \leq n.$$

Let $(Y_{a,j})_{j=1, a \in A}^\infty$ be independent copies of this process and define

$$(1) \quad X_a = \sum_{j=1}^{\infty} (\Gamma_j)^{-1/p} Y_{a,j}.$$

Then X_a is a (vector valued) p -stable process with $E\|X_a\|_U \geq (c_p/2)ST_p(U)$ ($c_p > 0$ depends only on p). Indeed, for each $\xi \in U^*$ and $a \in A$,

$$\xi(X_a) = \sum_{j=1}^{\infty} \Gamma_j^{-1/p} \xi(Y_{a,j}),$$

thus $\xi(X_a)$ is a symmetric p -stable random variable with parameter

$$C_p \cdot E|\xi(Y_{a,j})|^p = C_p \cdot \frac{1}{n} \sum_{i=1}^n |\xi(x_i)|^p.$$

The random variable $\xi(\sum_{i=1}^n \theta_i x_i)$ is also a symmetric p -stable with parameter proportional to $\sum_{i=1}^n |\xi(x_i)|^p$. It follows that

$$(2) \quad E\|X_a\| = \frac{c_p}{n^{1/p}} E \left\| \sum_{i=1}^n \theta_i x_i \right\| \geq \frac{c_p}{2} ST_p(U).$$

Next we replace the process $(X_a)_{a \in A}$ by the process

$$(3) \quad \tilde{X}_a = \sum_{j=1}^{\infty} j^{-1/p} Y_{a,j}, \quad a \in A.$$

We want to show that $E\|\tilde{X}_a\|$ is still large. This can be done as in [8]:

$$(4) \quad \begin{aligned} |E\|X_a\| - E\|\tilde{X}_a\|| &\leq \sum_{j=1}^{\infty} E|\Gamma_j^{-1/p} - j^{-1/p}| E\|Y_{a,j}\| \\ &= \sum_{j=1}^{\infty} E|\Gamma_j^{-1/p} - j^{-1/p}| E|y_a| \\ &\leq \sum_{j=1}^{\infty} E|\Gamma_j^{-1/p} - j^{-1/p}| = \phi < \infty \end{aligned}$$

and we get that, if $ST_p(U) > 4 \cdot c_p^{-1} \cdot \phi$,

$$(5) \quad E\|\tilde{X}_a\| \geq (c_p/4) ST_p(U).$$

If one tries to continue and follow [8] (trying to embed $(y_a)_{a \in A}$ into $[x_i]_{i=1}^n$) one runs into problems. We would like to get a good estimate on the probability that $\|\tilde{X}_a\|$ deviates by a certain amount from its expectation. The problem is that the martingale inequality used in [8] to do that (and other similar inequalities [9, 2]) uses the L_∞ norm of the martingale differences which in our case is $\approx (1/j^{1/p})\|y_a\|_\infty$ on which we have no control. We overcome this difficulty using the following lemma.

LEMMA 1. *Let $1 \leq p \leq 2$ and let Z be an m -dimensional subspace of $L_p(0, 1)$; then there exists a probability measure μ on $[0, 1]$ and a subspace $W \subseteq L_p(\mu)$ isometric to Z and such that*

$$(6) \quad \sup\{\|y\|_\infty; y \in W, \|y\|_{L_p(\mu)} = 1\} \leq \sqrt{2} m^{1/2} d(Z, l_2^m).$$

PROOF. Let $x_1, \dots, x_m$ be a basis in Z such that

$$C^{-1} \left(\sum_{i=1}^m a_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^m a_i x_i \right\|_{L_p(0,1)} \leq D \left(\sum_{i=1}^m a_i^2 \right)^{1/2}$$

for all $a_1, \dots, a_m \in \mathbf{R}$ where $C \cdot D = d(Z, l_2^m)$. Let μ be the probability measure

$$d_\mu = \left(\left(\sum_{i=1}^m x_i^2 \right)^{p/2} / \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_{L_p(0,1)}^p \right) dt.$$

Define $T: Z \rightarrow L_p(\mu)$ by

$$Tx = \frac{x}{\left(\sum_{i=1}^m x_i^2 \right)^{1/2}} \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_{L_p(0,1)}.$$

Then T is an isometry and if $x = \sum_{i=1}^m a_i x_i$ is of norm one in Z then, by Hölder's and Khinchine's inequalities and by the triangle inequality,

$$\begin{aligned}
 |Tx| &= \frac{\sum_{i=1}^m a_i x_i}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_{L_p(0,1)} \\
 &\leq \left(\sum_{i=1}^m a_i^2 \right)^{1/2} \cdot \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_{L_p(0,1)} \leq C \cdot \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_{L_p(0,1)} \\
 &\leq \sqrt{2} \cdot C \cdot \int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|_{L_p(0,1)} dt \\
 &\leq \sqrt{2} \cdot C \cdot D \cdot \sqrt{m} = \sqrt{2} \cdot m^{1/2} \cdot d(Z, l_2^m). \quad \square
 \end{aligned}$$

Returning to the discussion preceding Lemma 1, we may take now $\{y_a\}$ to be an α -net in the unit sphere of Z , an m -dimensional subspace of $L_p(\mu)$, assuming without loss of generality that $\|y_a\|_\infty \leq \sqrt{2} \cdot m^{1/2} \cdot d(Z, l_2^m) \leq \sqrt{2} \cdot m^{1/p}$ (see [3] and Remark 2 below). Proceeding as in [8] one can get that Z $(1 + \alpha)$ -embeds into U whenever $m \leq c_\alpha \cdot ST_p(U)^\beta$ where β depends on p and tends to ∞ as $p \rightarrow 1$. With some additional work, which we describe below, one can get a better estimate.

Let, as before, $(y_a)_{a \in A}$ be a set of norm one elements in $L_p(\mu)$ (μ a probability measure) and assume that $\|y_a\|_\infty \leq K < \infty$ for all $a \in A$. Let $(Y_a)_{a \in A}$, $(X_a)_{a \in A}$ and $(\tilde{X}_a)_{a \in A}$ be the processes defined above. We first want to modify $\tilde{X}_a$ by starting the summation $\sum j^{-1/p} Y_{a,j}$ at some large integer in such a manner that the expectation of the norm of this new process will remain of the same order of magnitude. First estimate $E\|\sum_{j=1}^l j^{-1/p} Y_{a,j}\|$ from above by the triangle inequality

$$(7) \quad E \left\| \sum_{j=1}^l j^{-1/p} Y_{a,j} \right\| \leq \sum_{j=1}^l j^{-1/p} E|y_a| \leq ql^{1/q}$$

$(1/p + 1/q = 1)$. Thus, if $l \leq ((\alpha c_p/4q)ST_p(U))^q$, then

$$\begin{aligned}
 \left| E \left\| \sum_{j=l}^\infty j^{-1/p} Y_{a,j} \right\| - E\|X_a\| \right| &\leq E \left\| \sum_{j=1}^l j^{-1/p} Y_{a,j} \right\| + |E\|\tilde{X}_a\| - E\|X_a\|| \\
 &\leq \frac{\alpha c_p}{4} ST_p(U) + \phi \quad (\text{by (4)}) \\
 &\leq \frac{\alpha c_p}{2} ST_p(U) \\
 &\leq \alpha E\|X_a\| \quad (\text{by (2)})
 \end{aligned}$$

as long as $\phi \leq (\alpha c_p/4)ST_p(U)$. For $l = [((\alpha c_p/4q)ST_p(U))^q]$, put

$$(8) \quad S_a = \sum_{j=l}^\infty j^{-1/p} Y_{a,j}.$$

Then

$$(9) \quad (1 - \alpha)E\|X_a\| \leq E\|S_a\| \leq (1 + \alpha)E\|X_a\|.$$

In particular,

$$(10) \quad E\|S_a\| \geq (c_p/4)ST_p(U).$$

We now want to show that the function $\|S_a\|$ is well concentrated near its expectation. We do it using a martingale inequality.

Let $\mathcal{F}_i$, $i = l-1, l, \dots$, be the σ -field generated by $Y_{a,1}, \dots, Y_{a,i}$, let E_i be the conditional expectation operator with respect to this σ -field and let $d_i = E_i\|S_a\| - E_{i-1}\|S_a\|$ be the martingale difference sequence generated by $\|S_a\|$. Then

$$(11) \quad \sum_{i=l}^{\infty} d_i = \|S_a\| - E\|S_a\|$$

and

$$(12) \quad i^{1/p} \cdot |d_i| \leq \|Y_{a,i}\| + E\|Y_{a,i}\| \leq |y_{a,i}| + E|y_{a,i}|.$$

In particular,

$$(13) \quad \|d_j\|_{\infty} \leq 2 \cdot j^{-1/p} \cdot K,$$

and

$$(14) \quad E_{j-1}d_j^2 \leq 4 \cdot j^{-2/p} \cdot E|y_a|^2 \leq 4 \cdot j^{-2/p} \cdot K^{2-p}$$

$$(\|y_a\|_2 \leq \|y_a\|_p^{p/2} \|y_a\|_{\infty}^{1-p/2} \leq K^{1-p/2}).$$

We now apply a martingale equality from [2]. This is a martingale version of Prokhorov's arc sinh inequality.

PROPOSITION 2. *Let $(d_i)_{i=1}^s$ be a mean zero bounded martingale difference sequence with respect to some increasing sequence $\mathcal{F}_1, \dots, \mathcal{F}_s$ of σ -fields, i.e., $E_{i-1}d_i = 0$, $i = 1, \dots, s$, where E_i is the conditional expectation operator with respect to $\mathcal{F}_i$. Let $\delta_i = \|d_i\|_{\infty}$, $M \geq \max_{1 \leq i \leq s} \delta_i$, $\eta_i^2 = \|E_{i-1}d_i^2\|_{\infty}$ and $\sigma^2 \geq \sum_{i=1}^s \eta_i^2$. Then, for all $t \geq 0$,*

$$P\left(\left|\sum_{i=1}^s d_i\right| \geq t\right) \leq 2 \exp\left(\frac{-t}{2M} \operatorname{arc sinh} \frac{Mt}{2\sigma^2}\right).$$

In our case one can choose

$$M = 2 \cdot l^{-1/p} \cdot K \quad \text{and} \quad \sigma^2 = 4 \cdot p \cdot K^{2-p}/(2-p)(l-1)^{2/p-1}$$

to get

$$P(\|\|S_a\| - E\|S_a\|\| > t) \leq 2 \exp\left(\frac{-t \cdot l^{1/p}}{4K} \operatorname{arc sinh}(c \cdot t \cdot K^{p-1} l^{-1/q})\right)$$

for some absolute constant c . Taking $t = \alpha E\|X_a\| \geq (c_p \alpha/2)ST_p(U)$, $K = \sqrt{2} m^{1/p}$ and $l = [((\alpha c_p/4q)ST_p(U))^q]$ we get that, for some absolute c ,

$$(15) \quad P((1-\alpha)E\|X_a\| \leq \|S_a\| \leq (1+\alpha)E\|X_a\|) \\ \geq 1 - 2 \exp\left(-\frac{c \alpha^q c_p^q \cdot ST_p(U)^q}{q^{q/p} \cdot m^{1/p}} \operatorname{arc sinh}(c \cdot q \cdot m^{1/q})\right).$$

We now finish the proof of Theorem 1 in the standard way. Let $\{y_a\}_{a \in I}$ be an α -net in the sphere of Z (an m -dimensional subspace of $L_p(\mu)$) with

$$\sup\{\|z\|_\infty; \|z\|_p = 1\} \leq \sqrt{2} m^{1/p}$$

with cardinality $\leq e^{2m/\alpha}$ (see e.g. [7]). Then

$$\begin{aligned} P((1 - \alpha)E\|X_a\| \leq \|S_a\| \leq (1 + \alpha)E\|X_a\| \text{ for all } a \in I) \\ \geq 1 - 2 \exp(2 \cdot m \cdot \alpha^{-1} - \delta(p, \alpha)(ST_p(U))^q \cdot m^{-1/p} \cdot \log m). \end{aligned}$$

Thus if $(ST_p(U))^q \geq \eta(p, \alpha) \cdot m^{1+1/p} \cdot (\log m)^{-1}$ for a suitable $\eta(p, \alpha) < \infty$, then there exists a w (in the probability space the S_a are defined on) such that

$$(1 - \alpha)E\|X_a\| \leq \|S_a(w)\| \leq (1 + \alpha)E\|X_a\| \quad \text{for all } a \in I.$$

Since $E\|X_a\|$ is a constant multiple of $\|y_a\| = 1$, it follows (see e.g. [7, Lemma 4.1]) that the map $y_a \rightarrow S_a(w)/E\|X_a\|$ extends to a linear map $T: Z \rightarrow U$ with

$$\frac{1 - 3\alpha}{1 - \alpha}\|y\| \leq \|Ty\| \leq \frac{1 + \alpha}{1 - \alpha}\|y\|.$$

This concludes the proof of Theorem 1. $\square$

Proof of Theorem 2. First notice that if $\sup\{\|z\|_\infty; \|z\|_1 = 1\} \leq \sqrt{2} \cdot m$ (as we may assume) then $\|z\|_1 \leq \|z\|_p \leq (1 + \alpha)\|z\|_1$ as long as $1 < p$ and $1 - 1/p \leq \delta(\alpha)/\log m$ for some $\delta(\alpha) > 0$. So we may assume that $Z \subseteq L_p$ for such a p . We now proceed as in the proof of Theorem 1 until we arrive at (15) in which the need arises to evaluate from below c_p and $(ST_p(l_1^n))^q$, for p as above.

Now, $(ST_p(l_1^n))^q = n$ for each $p > 1$. To evaluate c_p in the inequality

$$E\|X_a\| \geq \frac{c_p}{n^{1/p}} E\left\| \sum_{i=1}^n \theta_i e_i \right\|$$

(e_i being the unit vector basis in l_1^n) notice that $n^{-1/p} E\|\sum_{i=1}^n \theta_i e_i\| = n^{1/q}$ which is going to be bounded as long as n is smaller than a (bounded) power of m , so that we need to evaluate $E\|X_a\|$ from below. Let v be defined by $P(v = \pm x_i) = 1/2n$, $1 \leq i \leq n$, with v independent of y_a . Then Y_a has the same distribution as $y_a v$ and $\|v\| = 1$. Hence (see Lemma 4 and Proposition 5 of [5]),

$$E\|X_a\| = E\left\| \sum_{j=1}^\infty \Gamma_j^{-1/p} y_{a,j} v_j \right\| = \|y_a\|_p E\left\| \sum_{j=1}^\infty \Gamma_j^{-1/p} v_j \right\|$$

and, by the contraction principle,

$$\begin{aligned} E\|X_a\| &\geq E\Gamma_1^{-1/p} = \int_0^\infty u^{-1/p} e^{-u} du \\ &\geq e^{-1} \int_0^1 u^{-1/p} du = e^{-1} \frac{1}{1 - 1/p} \geq e^{-1} \cdot \delta^{-1}(\alpha) \cdot \log m. \end{aligned}$$

Plugging these estimates back in (15) and continuing as in the proof of Theorem 1 we get the desired result. $\square$

Remarks. (1) In Theorem 1 for the case $U = l_1^n$, if one is interested only in a uniform embedding (i.e. $T: X \rightarrow l_1^n$ with $\|T\| \|T|_{TX}^{-1}\| \leq K$ for some K depending only on p) rather than $(1 + \alpha)$ -embedding then a better result can be proved: An m -dimensional subspace X of $L_p(0, 1)$, $1 < p < 2$, K embeds into l_1^n as long as $n \geq c(p)m^{4/3}$. To prove this one first embed X into l_r^n for an appropriate $0 < r < 1$ and then use a theorem of Maurey [6] to show that this embedding, followed by a change of density, acts actually into l_1^n . Since we believe that the right result should be $n \geq c(p) \cdot m$ we do not bring in the details here.

(2) Lemma 1 implies Lewis' theorem [3] estimating the distance of an m -dimensional subspace of L_p , $1 \leq p < 2$, to l_2^m except for a constant. Indeed, given an m -dimensional subspace Z of $L_p(0, 1)$, $1 \leq p < 2$, we may assume by Lemma 1 that

$$\sup \{ \|z\|_\infty; z \in Z, \|z\|_p = 1 \} \leq \sqrt{2} m^{1/2} d(Z, l_2^m).$$

Then, for $z \in Z$,

$$\|z\|_p \leq \|z\|_2 \leq \|z\|_p^{p/2} \cdot \|z\|_\infty^{1-p/2} \leq (\sqrt{2} \cdot m^{1/2} d(Z, l_2^m))^{1-p/2} \|z\|_p.$$

So

$$d(Z, l_2^m) \leq (\sqrt{2} \cdot m^{1/2} \cdot d(Z, l_2^m))^{1-p/2},$$

and

$$d(Z, l_2^m) \leq (2 \cdot m)^{1/p-1/2}.$$

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