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THREE-SPACE PROBLEM FOR LOCALLY UNIFORMLY ROTUND RENORMINGS OF BANACH SPACES

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ABSTRACT. If Y is a subspace of a real Banach space X such that X/Y admits an equivalent LUR norm, then X admits an equivalent LUR (strictly convex) norm provided Y also does.

1. Introduction. It may happen that a Banach space with quite complicated structure may possess nice factors through nice subspaces (see e.g. [**4**, **6**, **7**]). Thus the question of what properties are shared by the whole space X, if satisfied in both $Y \subset X$ and X/Y, is of some interest. Concerning such properties linked with renorming theory, it is known that such a property, for instance, is being isomorphic to a uniformly convex space [**4**] while it is not the case for the property of being isomorphic to Hilbert space [**4**, **7**], nor for the property to be weakly compactly generated (see e.g. [**3**]). Recently M. Talagrand proved that this also is not the case for the property of the space admitting an equivalent Gateaux smooth norm [**11**].

A norm $|\cdot|$ of a Banach space X is called locally uniformly rotund (LUR) if $\lim|x_n - x| = 0$ for each x_n, $x \in X$, for which $\lim 2|x|^2 + 2|x_n|^2 - |x + x_n|^2 = 0$. The result of this paper originated from a more detailed study of the geometry in Day's construction of an LUR norm on $c_0(\Gamma)$ (see [**10**]) and, mainly, of its extension to spaces with transfinite Schauder bases in [**12**]. Some arguments of [**5**] are used here too.

Partial results connected with our results, namely those for the cases where either Y or X/Y are separable, were proved in [**1**] and [**5**].

The main ideas for this paper arose from discussions among the authors at the Winter School of Abstract Analysis (Czechoslovakia, January 1983) and the paper was prepared while the last named author was a member of Sonderforschungsbereich 72 der Universität Bonn, in fall 1983.

2. Result.

THEOREM. *Let X be a real Banach space and Y be such a subspace of X such that X/Y admits an equivalent LUR norm. Then X admits an equivalent LUR* (*strictly convex*) *norm provided that the subspace Y also does.*

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PROOF. (Depends heavily on that in [12].) We shall consider only the case of LUR; the other case can be dealt with similarly (see [5]). Let $\|\cdot\|$ be an equivalent norm on X, the restriction of which to Y is LUR. (For a simple construction of such a norm see e.g. [5].) Furthermore, let $|\cdot|$ denote an equivalent LUR norm on X/Y which is greater than or equal to the factor norm $|\cdot|_1$ on X/Y given by $\|\cdot\|$. Denote by $S_1 = \{\hat{x} \in X/Y, |\hat{x}| = 1\}$, where $\hat{x}$ means the element of X/Y given by x.

Let B: $X/Y \to X$ denote the Bartle-Graves continuous selection map (i.e. $B\hat{x} \in \hat{x}$ —see e.g. [2, p. 86]). For each $\hat{a} \in S_1 \subset X/Y$, let $f_{\hat{a}} \in X^*$ be such that $f_{\hat{a}}(B\hat{a}) = 1$, $\|f_{\hat{a}}\| = |\hat{a}|_1^{-1}$, $f_{\hat{a}} = 0$ on Y, and let, for $x \in X$, $P_{\hat{a}}(x) = f_{\hat{a}}(x) \cdot (B\hat{a})$. Now, for each $\hat{a} \in S_1 \subset X/Y$ and each positive integer k, define the following function $\Phi_{k,\hat{a}}$ on X:

$$\Phi_{k,\hat{a}}(x) = |\hat{x} + \hat{a}|^2 + k^{-1}\left(1 + \|P_{\hat{a}}\|\right)^{-2}\|x - P_{\hat{a}}(x)\|^2, \quad \text{for } x \in X.$$

Furthermore, let

$$\Phi_k(x) = \sup\{\Phi_{k,\hat{a}}(x), \hat{a} \in S_1 \subset X/Y\}, \quad \text{for } x \in X,$$

and

$$\Phi(x) = \|x\|^2 + |\hat{x}|^2 + \sum_k 2^{-k}\Phi_k(x), \quad \text{for } x \in X.$$

Finally, let $|||\cdot|||$ be the Minkowski functional of the set $\{x \in X\text{: } \Phi(x) + \Phi(-x) \leqslant 4\}$.

The functions $\Phi_{k,\hat{a}}(x)$ will be used to transfer the LUR property of the norms $\|\cdot\|$ on Y and $|\cdot|$ on X/Y to the whole space X.

It is easy to see that $|||\cdot|||$ is an equivalent norm on X. We now show that it is LUR. To do this suppose that, for some $\varepsilon > 0$, $x \in X$ and sequence $\{x_n\}$ such that

$$(1) \qquad |||x||| = 1 = |||x_n|||, \quad \lim|||x + x_n||| = 2 \quad \text{and} \quad \|x - x_n\| > \varepsilon > 0$$

and find a contradiction.

Because of the uniform continuity of the function $\Phi_0(x) = \Phi(x) + \Phi(-x)$ on bounded sets on X, we have from (1), that

$$\Phi_0(x) = \Phi_0(x_n) = 1, \qquad \lim \Phi_0((x + x_n)/2) = 1$$

and thus

$$\tfrac{1}{2}\Phi_0(x) + \tfrac{1}{2}\Phi_0(x_n) - \Phi_0((x + x_n)/2) \to_n 0,$$

and, from a convexity argument,

$$(2) \qquad \tfrac{1}{2}\Phi(x) + \tfrac{1}{2}\Phi(x_n) - \Phi((x + x_n)/2) \to_n 0.$$

Again by convexity, (2) implies that

$$(3) \qquad \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x_n\|^2 - \|(x + x_n)/2\|^2 \to_n 0,$$

$$(4) \qquad \tfrac{1}{2}|\hat{x}|^2 + \tfrac{1}{2}|\hat{x}_n|^2 - |(\hat{x} + \hat{x}_n)/2|^2 \to_n 0,$$

and, for each k,

$$(5) \qquad \tfrac{1}{2}\Phi_k(x) + \tfrac{1}{2}\Phi_k(x_n) - \Phi_k((x + x_n)/2) \to_n 0,$$

and

(6) $$K = \sup\{\|x_n\|\} < \infty.$$

First, if $x \in Y$, then $\hat{x} = 0$ and, from (5), we have $\lim|\hat{x}_n| = 0$. Thus there are $x'_n \in Y$, with $\lim\|x_n - x'_n\| = 0$. Then we have by (3) that

$$2\|x\|^2 + 2\|x'_n\|^2 - \|x + x'_n\|^2 \to 0$$

and, since $\|\cdot\|$ is LUR on Y, $\lim\|x'_n - x\| = 0$, which contradicts (1).

If $x \notin Y$, let $t = |\hat{x}|^{-1} > 0$. Write $tx = y_0 + s_0$, $s_0 = B(t\hat{x})$, $y_0 \in Y$.

From LUR of $\|\cdot\|$ on Y, there is a $\delta \in (0, \frac{1}{2})$ such that whenever $y \in Y$, $\|y - y_0\| \leqslant \delta$, $z \in Y$, $\frac{1}{2}\|y\|^2 + \frac{1}{2}\|z\|^2 - \|(y + z)/2\|^2 \leqslant \delta$, then

(7) $$\|y - z\| < \varepsilon t/2.$$

Let $\delta_1 \in (0, \delta)$ be such that whenever $y \in X$, $|\hat{y} - t\hat{x}| < \delta_1$, then

(8) $$\|P_{\hat{y}}\| < \|s_0\| \cdot |\hat{s}_0|_1^{-1} + 1.$$

Put

$$\delta_2 = \min\left\{10^{-1}(tK + 1)^{-1}\left(\|s_0\| \cdot |\hat{s}_0|_1^{-1} + 2\right)^{-2}\delta,\ \varepsilon t/8,\ \delta_1\right\}.$$

From LUR of $|\cdot|$ of X/Y, choose $\delta_3 > 0$ such that if $\hat{a} \in S_1 \subset X/Y$, $|\hat{x} + \hat{a}|^2 \geqslant (t^{-1} + 1)^2(1 - 4\delta_3)$, then

(9) $$|t\hat{x} - \hat{a}| < \delta_2 \quad \text{and} \quad \|B(t\hat{x}) - B(\hat{a})\| < \delta_2$$

(see e.g. [**8**, p. 343]). Finally choose, in the definition of our norm, an integer k such that

(10) $$k > \delta_3^{-1}K^{-2}$$

and fix this k until the end of the proof.

From (5) and (4) we have that

(11) $$c_n \equiv \tfrac{1}{2}\Phi_k(x) + \tfrac{1}{2}\Phi_k(x_n) - \Phi_k((x + x_n)/2) \to_n 0 \quad \text{and} \quad \lim|\hat{x}_n - \hat{x}| = 0.$$

Let $\hat{a}_n \in S_1 \subset X/Y$ be such that

$$d_n \equiv \Phi_k((x + x_n)/2) - \Phi_{k,\hat{a}_n}((x + x_n)/2) \to_n 0.$$

Then

$$c_n \geqslant \tfrac{1}{2}\Phi_{k,\hat{a}_n}(x) + \tfrac{1}{2}\Phi_{k,\hat{a}_n}(x_n) - \Phi_{k,\hat{a}_n}((x + x_n)/2) - d_n = b_n - d_n$$

for some nonnegative b_n, and thus, since $\lim c_n = \lim d_n = 0$, we have that $\lim b_n = 0$ as well. Therefore

$$\begin{aligned} b_n = {}& \frac{1}{2}|\hat{x} + \hat{a}_n|^2 + \frac{1}{2k}\left(1 + \|P_{\hat{a}_n}\|\right)^{-2}\|x - P_{\hat{a}_n}(x)\|^2 \\ & + \frac{1}{2}|\hat{x}_n + \hat{a}_n|^2 + \frac{1}{2k}\left(1 + \|P_{\hat{a}_n}\|\right)^{-2}\|x_n - P_{\hat{a}_n}(x_n)\|^2 \\ & - \left|\frac{\hat{x} + \hat{x}_n}{2} + \hat{a}_n\right|^2 - \frac{1}{k}\left(1 + \|P_{\hat{a}_n}\|\right)^{-2} - \left\|\frac{x + x_n}{n} - P_{\hat{a}_n}\left(\frac{x + x_n}{2}\right)\right\|^2 \to_n 0 \end{aligned}$$

and by the convexity argument,

$$(12)\qquad \begin{aligned}&\left(1+\|P_{\hat a_n}\|\right)^{-2}\frac{1}{2}\|x-P_{\hat a_n}(x)\|^2+\frac{1}{2}\|x_n-P_{\hat a_n}(x_n)\|^2\\ &-\left\|\frac{x+x_n}{2}-P_{\hat a_n}\left(\frac{x+x_n}{2}\right)\right\|^2\to_n 0\end{aligned}.$$

We now show that beginning with some index n_0, we have that

$$(13)\qquad |\hat x+\hat a_n|^2\geqslant (t^{-1}+1)^2(1-4\delta_3).$$

For, if it were not the case, then taking $\hat a_n'=t^{-1}\hat x$ we would have, for infinitely many n's,

$$|\hat x+\hat a_n'|^2>|\hat x+\hat a_n|^2+4\delta_3.$$

Then for these n's we would have, because of the convexity

$$\begin{aligned}c_n\geqslant{}&\frac{1}{2}|\hat x+\hat a_n'|^2+\frac{1}{2k}\left(1+\|P_{\hat a_n'}\|\right)^{-2}\|\hat x-P_{\hat a_n'}(x)\|^2\\
&+\frac{1}{2}|\hat x_n+\hat a_n|^2+\frac{1}{2k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\|x_n-P_{\hat a_n}(x_n)\|^2\\
&-\left|\frac{\hat x+\hat x_n}{2}+\hat a_n\right|^2-\frac{1}{k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\left\|\frac{x+x_n}{2}-P_{\hat a_n}\left(\frac{x+x_n}{2}\right)\right\|^2-d_n\\
={}&\frac{1}{2}|\hat x+\hat a_n'|^2-\frac{1}{2}|\hat x+\hat a_n|^2\\
&+\frac{1}{2k}\left(1+\|P_{\hat a_n'}\|\right)^{-2}\|x-P_{\hat a_n'}(x)\|^2-\frac{1}{2k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\|x-P_{\hat a_n}x\|^2\\
&+\frac{1}{2}|\hat x+\hat a_n|^2+\frac{1}{2}|\hat x_n+\hat a_n|^2-\left|\frac{\hat x+\hat x_n}{2}+\hat a_n\right|^2\\
&+\frac{1}{2k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\|x-P_{\hat a_n}(x)\|^2+\frac{1}{2k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\|x_n-P_{\hat a_n}(x_n)\|^2\\
&-\frac{1}{k}\left(1+\|P_{\hat a_n}\|\right)^{-2}\left\|\frac{x+x_n}{2}-P_{\hat a_n}\left(\frac{x+x_n}{2}\right)\right\|^2-d_n\\
\geqslant{}&\delta_3-d_n\end{aligned}$$

which contradicts $c_n\to 0$, $d_n\to 0$. Therefore, beginning with some index n_0, we have that $|\hat x+\hat a_n|^2\geqslant(t^{-1}+1)^2(1-4\delta_3)$ and hence by (9) and (8) we have that

$$(14)\qquad |\hat a_n-t\hat x|<\delta_2\leqslant\delta_1,\quad \|B\hat a_n-B(t\hat x)\|<\delta_2 \text{ and } \|P_{\hat a_n}\|<\|s_0\|\cdot|\hat s_0|_1^{-1}+1.$$

Thus, by (12), for sufficiently large $n\geqslant n_0$, we have that

$$(15)\qquad \frac{1}{2}\|tx-P_{\hat a_n}(tx)\|^2+\frac{1}{2}\|tx_n-P_{\hat a_n}(tx_n)\|^2-\left\|\frac{tx+tx_n}{2}-P_{\hat a_n}\left(\frac{tx+tx_n}{2}\right)\right\|^2<\delta_2$$

and

$$(16)\qquad |\hat a_n-\hat s_0|<\delta_2$$

and

(17) $$|t\hat{x}_n - t\hat{x}| < \delta_2$$

(use (4) together with LUR of $|\cdot|$ on X/Y). Fix such an n until the end of our proof.

Now choose an element $z_n \in \hat{a}_n$ such that

(18) $$\|z_n - tx\| < \delta_2$$

(use (16)), and an element $x'_n \in \hat{a}_n$ such that

(19) $$\|x'_n - tx_n\| < 2\delta_2$$

(use (16) and (17)).

Setting $a_n = B\hat{a}_n$, write $x'_n = a_n + u_n$, $u_n \in Y$; $z_n = a_n + v_n$, $v_n \in Y$. Then

$$x'_n - P_{\hat{a}_n}\hat{x}'_n = u_n, \qquad z_n - P_{\hat{a}_n}z_n = v_n$$

and, since $tx = s_0 + y_0$ and $z_n = a_n + v_n$, we have that

$$\begin{aligned}\|v_n - y_0\| &\leqslant \|tx - z_n\| + \|s_0 - a_n\| \\ &= \|tx - z_n\| + \|B(t\hat{x}) - B\hat{a}_n\| \leqslant 2\delta_2 \leqslant \delta_1,\end{aligned}$$

(use (14)). Moreover, we have (use (15))

$$\begin{aligned}&\frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u_n\|^2 - \left\|\frac{u_n + v_n}{2}\right\|^2 \\ &= \frac{1}{2}\left\|z_n - P_{\hat{a}_n}(z_n)\right\|^2 + \frac{1}{2}\left\|x'_n - P_{\hat{a}_n}(x'_n)\right\|^2 - \left\|\frac{x'_n + z_n}{2} - P_{\hat{a}_n}\left(\frac{x'_n + z_n}{2}\right)\right\|^2 \\ &\leqslant \frac{1}{2}\left\|tx - P_{\hat{a}_n}(tx)\right\|^2 + \frac{1}{2}\left\|tx_n - P_{\hat{a}_n}(tx_n)\right\|^2 - \left\|\frac{tx + tx_n}{2} - P_{\hat{a}_n}\left(\frac{tx + tx_n}{2}\right)\right\|^2 \\ &\quad + \frac{1}{2}\left\|z_n - tx - P_{\hat{a}_n}(z_n - tx)\right\|\left(\left\|z_n - P_{\hat{a}_n}(z_n)\right\| + \left\|tx - P_{\hat{a}_n}(tx)\right\|\right) \\ &\quad + \frac{1}{2}\left\|x'_n - tx_n - P_{\hat{a}_n}(x'_n - tx_n)\right\|\left(\left\|x'_n - P_{\hat{a}_n}(x'_n)\right\| + \left\|tx_n - P_{\hat{a}_n}(tx_n)\right\|\right) \\ &\quad + \frac{1}{2}\left(\left\|\frac{z_n - tx}{2} - P_{\hat{a}_n}\left(\frac{z_n - tx}{2}\right)\right\| + \left\|\frac{x'_n - tx_n}{2} - P_{\hat{a}_n}\left(\frac{x'_n - tx_n}{2}\right)\right\|\right) \\ &\quad \cdot\left(\left\|z_n - P_{\hat{a}_n}(z_n)\right\| + \left\|x'_n - P_{\hat{a}_n}(x'_n)\right\| + \left\|tx - P_{\hat{a}_n}(tx)\right\| + \left\|tx_n - P_{\hat{a}_n}(tx_n)\right\|\right) \\ &\leqslant 10\delta_2(tK + 1)\left(\|s_0\||\hat{s}_0|_1^{-1} + 2\right)^2 \leqslant \delta\end{aligned}$$

(use (6), (18) and (19)). Therefore, by (7),

(20) $$\varepsilon/2 \geqslant \|u_n - v_n\| = \left\|x'_n - P_{\hat{a}_n}x'_n - z_n + P_{\hat{a}_n}z_n\right\| = \left\|x'_n - z_n\right\|.$$

Thus by (18), (19) and (20), we have

$$\begin{aligned}\|tx_n - tx\| &\leqslant \|tx_n - x'_n\| + \|x'_n - z_n\| + \|z_n - tx\| \\ &\leqslant 2\delta_2 + \frac{\varepsilon t}{2} + \delta_2 = 3\delta_2 + \frac{\varepsilon t}{2} < \frac{3\varepsilon t}{8} + \frac{\varepsilon t}{2} < \varepsilon t.\end{aligned}$$

Thus $\|x_n - x\| < \varepsilon$, a contradiction and the proof is finished.

We end the paper with the following apparently open problem: Let X be a Banach space and Y be such a subspace of X that both Y and X/Y admit equivalent strictly convex norms. Must X admit an equivalent strictly convex norm?

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