ON THE RIESZ-TRANSFORMS ALONG SURFACES IN R^3

A. EL KOHEN

ABSTRACT. For $x = (x_1, x_2, x_3)$ in R^3 , $t = (t_1, t_2)$ in R^2 , $|t| = (t_1^2 + t_2^2)^{1/2}$ and $a > 0$, we define

$$R_a f(x) = \int_{R^2} f(x_1 - t_1, x_2 - t_2, x_3 - |t|^a) \frac{t_1}{|t|^3} dt.$$

The transformation R_a can be thought of as a Riesz-transform along the surface $(t_1, t_2, |t|^a)$ in R^3 . Our purpose here is to show that, for all $a > 0$, the operator R_a is bounded on L^p for $6/5 < p < 6$.

Singular integrals along lower dimensional nonaffine varieties have been studied by several authors. See for example [3, 5, 6]. In the proof of our result, we use techniques similar to the ones used in [1, 2].

0. A simple calculation gives $R_a \widehat{f} = m_a \hat{f}$ where for $u = (u_1, u_2, u_3)$ we have

$$m_a(u) = \frac{2\pi u_1}{|u'|} \int_0^{+\infty} J'_0(r) e^{-ir^a} \frac{u_3}{|u'|^a} \frac{dr}{r}$$

with $u' = (u_1, u_2)$.

From the identity $J'_0 = -J_1$ and the well-known estimates on Bessel functions [4] one deduces that $g(t) = \int_0^{+\infty} J'_0(r) e^{-ir^a} dr/r$ is a bounded function, consequently $m_a(u)$ is also a bounded function. Thus, for all $a > 0$, the operator R_a is bounded on L^2 .

THEOREM. *For all $a > 0$, the operator R_a is bounded on L^p for $6/5 < p < 6$.*

We let p, q be smooth functions on the real line, $p + q = 1$, $p(t) = 1$ for $-1/2 \leq t \leq 1/2$ and $p(t) = 0$ for $t \leq -1$ or $t \geq 1$.

To prove our result, it is clear that all we need to show is that the convolution operator corresponding to the multiplier

$$n_a(u) = \int_0^{+\infty} J'_0(r) e^{-ir^a} \frac{u_3}{|u'|^a} \frac{dr}{r}$$

is bounded on L^p for $6/5 < p < 6$.

We let

$$F_{xy}(t) = x(t) \int_0^{+\infty} y(r) J'_0(r) e^{-ir^a t} \frac{dr}{r},$$

$x, y = p, q$, and shall estimate separately each of the multipliers

$$n_{a,xy}(u) = F_{xy} \left(\frac{u_3}{|u'|^a} \right), \quad x, y = p, q.$$

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1. It is easily seen that $n_{a,pp}$ verifies

$$|u^\alpha \partial_u^\alpha n_{a,pp}(u)| \leq C_\alpha$$

for all triples α of nonnegative integers, from which one deduces that the corresponding convolution operator is bounded on L^p for all $1 < p < +\infty$.

For $F(t)$ a function of one variable t , we let

$$K_-(\gamma) = \int_0^{+\infty} F(-t)t^{-i\gamma} \frac{dt}{t}, \quad K_+(\gamma) = \int_0^{+\infty} F(t)t^{-i\gamma} \frac{dt}{t}.$$

Let T_a be the convolutor corresponding to the multiplier $F(u_3/|u'|^a)$. Then

$$\begin{aligned} T_a f(x) &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} K_-(\gamma) U_\gamma V_{a\gamma} (I + iH_3) f(x) d\gamma \\ &\quad + \frac{1}{4\pi} \int_{-\infty}^{+\infty} K_+(\gamma) U_\gamma V_{a\gamma} (I - iH_3) f(x) d\gamma \end{aligned}$$

where H_3 is the Hilbert transform in the variable x_3 , U_γ (resp. V_γ) the convolutor corresponding to the multiplier $|u_3|^{i\gamma}$ (resp. $|u'|^{-i\gamma}$).

Consequently we need to estimate operators of the form

$$T'_a = \int_{-\infty}^{+\infty} K_\pm(\gamma) U_\gamma V_{a\gamma} d\gamma.$$

2. After some simple estimates, for $t \sim 0$, $F_{pq}(t)$ is equivalent to the sum of two terms of the form

$$cp(t) + dp(t)|t|^{3/2a} e^{\pm i|t|^{-1/a}}$$

where c, d are constants.

The multiplier $p(u_3/|u'|^a)$ satisfies the same estimates as $n_{a,pp}(u)$ in 1; hence the corresponding convolutor is bounded on L^p for $1 < p < +\infty$. We then let $F(t) = p(t)|t|^{3/2a} e^{\pm i|t|^{-1/a}}$; the corresponding functions $K_\pm$ are given by

$$K_\pm(\gamma) = \int_0^{+\infty} p(\pm t) t^{3/2a} e^{\pm it^{-1/a}} t^{-i\gamma} \frac{dt}{t}.$$

The functions $K_\pm$ are easily seen to satisfy $|K_\pm(\gamma)| \leq (\text{const})/(1 + |\gamma|)^2$.

3. For F_{qp} we have $F_{qp}(t) \sim kq(t)|t|^{-1/a}$ for $t \sim \infty$ where k is a constant. If we let $F(t) = q(t)|t|^{-1/a}$, it is easily seen that the corresponding functions $K_\pm$ are bounded and rapidly decreasing at infinity.

4. From the estimates on Bessel functions [4] we have

$$F_{qq}(t) \sim q(t) \int_0^{+\infty} q(r) e^{\pm ir} e^{-ir^a t} \frac{dr}{r^{3/2}} \quad \text{for } t \sim \infty.$$

For $0 < a \leq 1$, repeated integrations by parts show that F_{qq} is a rapidly decreasing function. For $a > 1$, we let $r = s|t|^b$ with $b = 1/(1-a)$, then

$$\begin{aligned} F_{qq}(t) &\sim |t|^{-b/2} q(t) \int_0^{+\infty} q(s|t|^b) e^{i(\pm s - s^a \operatorname{sgn} t)|t|^b} \frac{ds}{s^{3/2}} \\ &\sim |t|^{-b/2} q(t) \int_0^{+\infty} q(s^{1/a}|t|^b) e^{-i \operatorname{sgn} t u|t|^b} \frac{du}{u^{1+1/2a}} \\ &= |t|^{1/2a} q(t) \int_{|t|/2}^{+\infty} e^{-i \operatorname{sgn} t v} q\left(\left(\frac{v}{t}\right)^{1/a}\right) \frac{dv}{v^{1+1/2a}}. \end{aligned}$$

From which one deduces that $F_{qq}(t) \sim q(t)e^{-it}/t^2$ for $t \sim \infty$. The corresponding functions $K_{\pm}$ are easily seen to satisfy $|K_{\pm}(\gamma)| \leq (\text{const})/(1+|\gamma|)^{5/2}$.

Note. The error terms in $F_{pq}-cp$, F_{qp} and F_{qq} have their corresponding functions $K_{\pm}(\gamma) = o(|\gamma|^{-2})$.

To finish the proof of our result, we use Lemmas 2, 4 of [1] to get

$$\|U_{\gamma}V_{a\gamma}\|_p \leq C_{a,p,d}(1+|\gamma|)^{3|1/p-1/2|}\text{Log}^3(1+|\gamma|).$$

Consequently,

$$\|T'_a\|_p \leq C_{a,p,d} \int_{-\infty}^{+\infty} |K_{\pm}(\gamma)|(1+|\gamma|)^{3|1/p-1/2|}\text{Log}^3(1+|\gamma|) d\gamma.$$

Using the estimates on $K_{\pm}$ given above, we see that for p satisfying $2-3|1/p-1/2| > 1$, T'_a is bounded on L^p for $6/5 < p < 6$.

Note. As a "Limiting" case, we define

$$Sf(x) = \int_{\mathbb{R}^2} f(x_1 - t_1, x_2 - t_2, x_2 - \text{Log}|t|) \frac{t_1}{|t|^3} dt.$$

We then have $Sf^{\wedge} = s\hat{f}$ where

$$s = \frac{4\pi u_1}{|u'|} |u'|^{-iu_3} \frac{\Gamma((1-iu_3)/2)}{2^{iu_3}(1+u_3^2)\Gamma((-1+iu_3)/2)}.$$

It is easily seen that s is a bounded function; hence S is a bounded operator on L^2 . We conjecture that S is bounded on L^p iff $p = 2$.

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FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUES, B.P. 1796-ATLAS-FÈS, MOROCCO