A CHARACTERIZATION THEOREM FOR BOUNDED STARSHAPED SETS IN THE PLANE

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ABSTRACT. Let S be a nonempty bounded set in R^2 . Then S is starshaped if and only if every 3 or fewer boundary points of S are clearly visible via S from a common point of S . The number 3 is best possible.

1. Introduction. We will need the following definitions from [1 and 2]. Let S be a subset of R^d . A point s in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of s such that $N \cap S$ is convex. If S fails to be locally convex at point q in S , then q is called a *point of local nonconvexity* (lnc point) of S . For points x and y in S , we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S . Similarly, for x in $\text{cl } S$ and y in S , x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees each point of $N \cap S$ via S . Finally, set S is called *starshaped* if and only if there is some point p in S such that p sees each point of S via S , and the set of all such points p is called the (convex) kernel of S , denoted $\text{ker } S$.

A well-known theorem of Krasnosel'skii [6] states that if S is a nonempty compact set in R^d , then S is starshaped if and only if every $d + 1$ points of S are visible from a common point of S . Moreover, a stronger result may be obtained by replacing points of S with boundary points of S . Unfortunately, these theorems fail for noncompact sets (examples in [3] reveal that no such theorem is possible without some modifications), and the problem of obtaining a Krasnosel'skii-type theorem to characterize noncompact starshaped sets remains an intriguing one. It is this type of characterization theorem which will be considered here. An important tool will be the concept of clearly visible, a notion which has appeared in papers by Stavrakas [7] and Falconer [4]. Recently (in [1 and 2]) analogues of the Krasnosel'skii theorem were obtained by replacing the concept of visible with that of clearly visible and by replacing points of S with points of local nonconvexity of S . Here this kind of replacement is used again to obtain the following result: Let S be a nonempty bounded set in R^2 . Then S is starshaped if and only if every 3 or fewer boundary points of S are clearly visible via S from a common point of S . The number 3 is best possible.

As in [1 and 2], the following familiar terminology will be used: $\text{Conv } S$, $\text{cl } S$, $\text{int } S$, $\text{rel int } S$, $\text{bdry } S$ and $\text{ker } S$ will denote the convex hull, closure, interior, relative interior, boundary, and kernel, respectively, for set S . $\text{lnc } S$ will be the set of points of local nonconvexity of S , and if S is convex, $\dim S$ will be the dimension of S . For distinct points x and y , $R(x, y)$ will represent the ray emanating from x through

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y , and $L(x, y)$ will be the line determined by x and y . The reader is referred to Valentine [8] for a discussion of these concepts.

2. The results. We will be concerned with the proof of the following theorem.

THEOREM 1. *Let S be a nonempty bounded set in R^2 . Then S is starshaped if and only if every 3 or fewer boundary points of S are clearly visible via S from a common point of S . The number 3 is best possible.*

PROOF. The necessity of the condition is obvious, so we need to establish only its sufficiency. Assume that set S satisfies the condition stated above. Then certainly S is a bounded connected set in R^2 , and every 3 or fewer lnc points of S are clearly visible via S from a common point. Hence we may repeat the construction in [2, Theorem A] to select a point p in S having the following properties: Point p is in $(\ker \text{cl } S) \cap S$, the corresponding set $A \equiv \{s: s \in S \text{ and } [p, s] \not\subseteq S\}$ is nowhere dense in S , and $A \subseteq \text{bdry } S$.

The proof of the theorem will be accomplished by considering cases determined by the arrangement of points in $(\text{int cl } S) \sim S$. For future reference, we remark that by [2, Lemma 3], every point in $(\text{int cl } S) \sim S$ lies in a nondegenerate segment in $(\text{int cl } S) \sim S$. There are three possibilities to consider.

Case 1. Assume that $(\text{int cl } S) \sim S = \emptyset$. We examine subcases determined by the dimension of $\ker \text{cl } S \neq \emptyset$.

Case 1a. In case $\dim(\ker \text{cl } S) = 0$, then point p defined above is the unique member of $\ker \text{cl } S$. We will show that $p \in \ker S$ as well. Let x be an arbitrary point of S to prove that $[p, x] \subseteq S$. If $x \in \text{int } S$, then since $A \subseteq \text{bdry } S$, the result is immediate. If $x \in \text{bdry } S$, assume on the contrary that there is some point in $(p, x) \sim S$ to reach a contradiction. Choose points z, z_0 on $[p, x]$ such that all points between p and z (if any exist) are in S , all points between z_0 and x (if any exist) are in S , and the corresponding lengths of $[p, z]$ and of $[z_0, x]$ are maximal. Then $z, z_0 \in \text{bdry } S$, and using the facts that $p \in \ker \text{cl } S$ and $(\text{int cl } S) \sim S = \emptyset$, it is easy to see that $[z, x] \subseteq \text{bdry } S$.

By a similar argument, z_0 is clearly visible via S only from points in one of the closed halfplanes determined by the line of $L \equiv L(p, x)$, say $\text{cl } L_1$. Let R_1 be a ray emanating from point p , $R_1 \subseteq L_1 \cup \{p\}$. If every boundary point of $\text{cl } S$ were clearly visible via $\text{cl } S$ from some point of $R_1 \sim \{p\}$, then using the fact that $\text{bdry } S$ is compact, methods used in [2] would yield a nondegenerate segment at p in R_1 such that every boundary point of $\text{cl } S$ could see each point of this segment via $\text{cl } S$. But then this segment would be in $\ker \text{cl } S$, impossible since $\ker \text{cl } S$ contains only point p . Thus, there exists a boundary point b_1 of $\text{cl } S$ not clearly visible via $\text{cl } S$ from any point of $R_1 \sim \{p\}$. Furthermore, we assert that b_1 and z_0 together are clearly visible via S only from points in one of the closed convex regions bounded by R_1 and L :

Suppose on the contrary that b_1 and z_0 are clearly visible via S from points s and t in opposite open halfplanes determined by the line of R_1 . Then $s, t \in \text{cl } L_1 \sim R_1$. Without loss of generality, assume that t and z_0 are on the same side of R_1 , s and z_0 on opposite sides. Certainly $s \notin L$ since z_0 is clearly visible via S from s . Consider in turn each of the following possible locations for point b_1 : $b_1 \in L_1$, $b_1 \in R(p, z_0)$, $b_1 \in R(z_0, p)$, $b_1 \in L_2$. In each case, select a convex neighborhood N of b_1 so that both s and t see via $\text{cl } S$ every point of $N \cap \text{cl } S$. Let $(s, z_0) \cap R_1 = \{p'\}$

and $(s, t) \cap R_1 = \{p''\}$. Since $s \notin L$, neither p' nor p'' is p . Let p_0 be the member of $\{p', p''\}$ closer to p . Using the facts that $[s, z_0] \subseteq \text{cl } S$ and $p \in \ker \text{cl } S$, it follows that points of (p, p_0) see via $\text{cl } S$ each point of $N \cap \text{cl } S$. However, b_1 is not clearly visible via $\text{cl } S$ from any point of $R_1 \sim \{p\}$, so this is impossible. We conclude that b_1 cannot be in any of these locations. Our supposition is false, and z_0 and b_1 together are clearly visible via S only from points in one closed convex region bounded by R_1 and L . Let T_1 denote this region and let $R_0 = L \cap \text{bdry } T_1$.

Continuing, let ray R_2 emanate from p and bisect the angle determined by R_0 and R_1 . Using the argument above, select a boundary point b_2 of $\text{cl } S$ such that b_2 is not clearly visible via $\text{cl } S$ from any point of $R_2 \sim \{p\}$. Then b_2 and z_0 are clearly visible via S only from points in one of the closed convex regions, say T_2 , bounded by R_2 and L . Furthermore, points b_1, b_2 and z_0 are clearly visible via S only from points in $T_1 \cap T_2$ which is bounded by R_2 and one of R_0, R_1 . By an obvious induction, we obtain a sequence of boundary points $\{b_n\}$ and corresponding sequences of rays $\{R_n\}$ and regions $\{T_n\}$ such that the following are true: For $n \geq 2$, point b_n is not clearly visible via $\text{cl } S$ from any point of $R_n \sim \{p\}$, ray R_n bisects the angle determined by $T_1 \cap \cdots \cap T_{n-1}$, and region $T_1 \cap \cdots \cap T_n$ is bounded by R_n and one of $R_0, R_1, \dots, R_{n-1}$. Observe that if $T_1 \cap \cdots \cap T_n$ is bounded by R_n and R_j for $j \geq 1$, then b_n, z_0, b_j are clearly visible via S only from points in $T_1 \cap \cdots \cap T_n$. Similarly, if $T_1 \cap \cdots \cap T_n$ is bounded by R_n and R_0 , then b_n and z_0 are clearly visible via S only from points in $T_1 \cap \cdots \cap T_n$.

The sequence of rays $\{R_n\}$ converges to some ray R . Let b be a limit point for $\{b_n\}$. Clearly, $b \in \text{bdry } S$ since $\{b_n\} \subseteq \text{bdry } S$. Now if $R = R_j$ for some $j \geq 1$, then it is easy to see that points b, b_j, z_0 can be clearly visible via S only from points on R_j . But b_j is not clearly visible via $\text{cl } S$ from any point of $R_j \sim \{p\}$ and hence b_j is not clearly visible via S from any point of $R_j \sim \{p\}$. Since z_0 is not clearly visible via S from p , points b, b_j, z_0 cannot be clearly visible via S from any common point. We have a contradiction, and $R \neq R_j$ for any $j \geq 1$.

Similarly, if $R = R_0$, then b and z_0 are clearly visible via S only from points on $R_0 \sim \{p\}$. Clearly such points lie on $R(p, z_0) \sim [p, z_0]$. Thus by our hypothesis for set S , every boundary point of S is clearly visible from a point of $R(p, z_0) \sim [p, z_0]$. By standard arguments, we obtain a nondegenerate segment at p in $\ker \text{cl } S$. This violates our hypothesis for Case 1a, and we conclude that $R \neq R_0$.

Since $R \neq R_k$ for any $k \geq 0$, there exist subsequences $\{R'_j\}, \{R''_j\}$ of R_k which converge to R and which lie in opposite closed half-planes $\text{cl } H_2, \text{cl } H_1$, respectively, determined by the line H of R . Let $\{b'_j\}$ and $\{b''_j\}$ denote the corresponding sets of boundary points of S . Since we may pass to convergent subsequences of $\{b'_j\}$ and $\{b''_j\}$, for convenience of notation, assume that $\{b'_j\}$ converges to point b' and that $\{b''_j\}$ converges to b'' . Furthermore, assume that H_1 and H_2 are labeled so that $z_0 \in H_1$. Then b' and z_0 are clearly visible via S only from points in $\text{cl } H_1 \cap \text{cl } L_1$, b'' and z_0 are clearly visible via S only from points in $\text{cl } H_2 \cap L_1$, and of course b', b'', z_0 are clearly visible via S only from points in $R \sim \{p\}$.

We will prove that every boundary point of S is clearly visible via S from a point of $R \sim \{p\}$: Let y be an arbitrary boundary point of S . Points y, b'', z_0 are clearly visible via S from some v , and by the remarks above, $v \in \text{cl } H_2 \cap L_1$. Similarly, points y, b', z_0 are clearly visible via S from some w , and $w \in \text{cl } H_1 \cap \text{cl } L_1 \sim [p, z_0]$. If v or w is on H , then certainly y is clearly visible from a point of $R \sim \{p\}$, the

desired result. If $v \in H_2$ and $w \in H_1$, a geometric argument shows that y is clearly visible via S from some point of $R \sim \{p\}$, and again we have our result.

Thus every boundary point of S is clearly visible via S from some point of $R \sim \{p\}$. Using the compactness of $\text{bdry } S$, standard arguments yield a nondegenerate segment in $\ker \text{cl } S$, again contradicting the fact that $\dim \ker \text{cl } S = 0$. Our original assumption must be false, $[p, x] \subseteq S$, and we conclude that $p \in \ker S$. In fact, it is easy to see that $\{p\} = \ker S$. This finishes the argument in Case 1a.

Case 1b. Suppose that $\dim \ker \text{cl } S = 1$. Let G be the line determined by $\ker \text{cl } S$. If for one open halfplane G_1 determined by G , every boundary point of S is clearly visible via S from a point of G_1 , then standard arguments similar to those in [2] yield a 2-dimensional subset of $\ker \text{cl } S$, impossible. Hence there exist boundary points b and c of S such that b is not clearly visible via S from any point in G_1 and c is not clearly visible via S from any point in G_2 .

We assert that $G \cap S$ is connected: Suppose not. Then $G \cap S$ has at least two components. For u, v distinct endpoints of different components of $G \cap S$, u and v are in $\text{bdry } S$ and are clearly visible via S only from points in $G_1 \cup G_2$. Since u, v, b are clearly visible from some common point of S , $G_2 \cap S \neq \emptyset$. A similar argument for u, v, c shows that $G_1 \cap S \neq \emptyset$. Using this information, it is easy to prove that every relatively interior point of $\ker \text{cl } S$ is in $\text{int cl } S$ and hence in S (since $(\text{int cl } S) \sim S = \emptyset$). We conclude that $\text{rel int}(\ker \text{cl } S)$ lies in one component K of $G \cap S$. Let y be an endpoint of K and let x be an endpoint of another component of $G \cap S$, with $x \notin \text{cl } K$. Since $(\text{int cl } S) \sim S = \emptyset$ and $x \notin \text{cl } K$, it is easy to show that for at least one of the open halfplanes G_1 and G_2 , say for G_1 , x fails to be clearly visible via S from any point of G_1 . Then x and c are clearly visible via S only from points on line G , and the boundary points x, y, c are clearly visible via S only from points on G . However, this implies that $(x, k] \subseteq S$ for some $k \in K$, so $(x, k] \subseteq K$, contradicting the fact that $x \notin \text{cl } K$. Our supposition must be false, and $G \cap S$ is indeed connected.

Using this result, we are ready to prove that S is starshaped. There are two possibilities to consider. In case $\text{rel int}(\ker \text{cl } S) \cap S$ contains some point q , we show that $q \in \ker S$: Select $s \in S$. If $s \in S \sim G$, then since $(\text{int cl } S) \sim S = \emptyset$, it is easy to see that $[q, s] \subseteq S$. Likewise, if s is in the connected set $G \cap S$, then $[q, s] \subseteq S$, so $q \in \ker S \neq \emptyset$.

In case $(\text{rel int}(\ker \text{cl } S)) \cap S = \emptyset$, let p be the point of $(\ker \text{cl } S) \cap S$ described in our opening paragraph. Certainly p must be an endpoint of $\ker \text{cl } S$. We show that $p \in \ker S$: Again let $s \in S$. If $s \in G \cap S$, then $[p, s] \subseteq S$, so assume that $s \in S \sim G$. Ray $R(p, s) \sim [p, s)$ meets $\text{bdry } S$ at some first point s' , and we note that if $s \neq s'$, then $[s, s') \subseteq S$. Using our hypothesis for set S , boundary points s', b, c are clearly visible via S from some common point t , and by our choice of b and c , t must belong to $G \cap S$, which is connected and disjoint from $\text{rel int}(\ker \text{cl } S)$. If $t = p$, then $[p, s') \subseteq S$ and so $[p, s] \subseteq S$. If $t \neq p$, then $[t, p] \subseteq G \cap S \subseteq \sim \text{rel int}(\ker \text{cl } S)$, and $t < p < r$ for $r \in \text{rel int}(\ker \text{cl } S)$. Thus $(p, s) \subseteq \text{int conv}\{t, s', \ker \text{cl } S\} \subseteq \text{int cl } S \subseteq S$, and $[p, s] \subseteq S$. Hence $p \in \ker S \neq \emptyset$, and we conclude that S is starshaped, finishing the argument in Case 1b.

Case 1c. In case $\dim \ker \text{cl } S = 2$, select any point q in $\text{int}(\ker \text{cl } S) \subseteq S$. For $s \in S$, it is easy to see that $[q, s] \subseteq \text{int cl } S$. Hence $[q, s] \subseteq S$, $q \in \ker S$, and S is starshaped.

This completes the proof in case $(\text{int cl } S) \sim S = \emptyset$. When $(\text{int cl } S) \sim S \neq \emptyset$, each of its points lies in a nondegenerate segment in $(\text{int cl } S) \sim S$, and the remaining cases will be determined by the arrangement of these segments.

Case 2. Assume that $(\text{int cl } S) \sim S$ contains two or more noncollinear segments. Let p be the point of $(\ker \text{cl } S) \cap S$ described in our opening paragraph, with $A = \{s: s \text{ in } S \text{ and } [p, s] \not\subseteq S\}$. By an easy adaptation of [2, Lemma 4], every boundary point of S is clearly visible from p via S , and p is unique. Then since $A \subseteq \text{bdry } S$, A must be empty and $\{p\} = \ker S$.

Case 3. Assume that $(\text{int cl } S) \sim S \neq \emptyset$ has all its segments on line J , and let T be a segment in $(\text{int cl } S) \sim S$ having maximal length. As usual, let p and A be the particular point and set defined in our opening paragraph. By [2, Lemmas 5 and 6], for one endpoint t of T , t is clearly visible via S only from points in some interval V at t , with $V \subseteq S$. Moreover, by the selection of point p in [2, Theorem 2], $p \in V$. For future reference, notice that $t \in \text{bdry } S$.

The following observation will be useful. For $u \in J \cap S$ and $v \in V \subseteq S$, we assert that $[u, v] \subseteq S$: If $u = p$, the result is immediate. If $u \neq p$, the ray $R(p, u) \sim [p, u]$ meets $\text{bdry } S$ at some first point u' , and if $u \neq u'$, then $[u, u'] \subseteq S$. Using our hypothesis for set S , u' and t are clearly visible via S from some point of V , so $[p, u'] \subseteq S$, $[p, u] \subseteq S$, and $[v, u] \subseteq S$. Our assertion is established.

Now if for every z in $\text{bdry } S \cap S$, $[p, z] \subseteq S$, then since $A \subseteq \text{bdry } S$, A must be empty, and $p \in \ker S$, finishing the argument in Case 3. Hence we assume that for some point w_0 in $\text{bdry } S \cap S$, $[p, w_0] \not\subseteq S$. By the observation above, $w_0 \notin J$. Furthermore, since all segments in $(\text{int cl } S) \sim S$ are on J and $[p, w_0] \not\subseteq S$, it is not hard to show that w_0 must be clearly visible via S only from points in one closed halfplane $\text{cl } M_1$ determined by line $L(p, w_0) \equiv M$. Then boundary points w_0 and t are clearly visible via S only from points in $(\text{cl } M_1) \cap V \sim \{p\} = M_1 \cap V \neq \emptyset$. Again by our hypothesis for set S , every boundary point of S is clearly visible via S from some point of $M_1 \cap V$, and using standard arguments, we obtain a nondegenerate interval I at p in $(\ker \text{cl } S) \cap M_1 \cap V \subseteq S$.

Let q be a point in $\text{relint } I$. We will show that $q \in \ker S$. Choose $s \in S$. If $s \in S \sim J$, since $(\text{int cl } S) \sim S \subseteq J$, it is easy to show that $[q, s] \subseteq S$. If $s \in J \cap S$, then by our previous observation, $[q, s] \subseteq S$. Hence $q \in \ker S \neq \emptyset$ and S is starshaped, completing the argument in Case 3 and finishing the proof of the theorem.

It is not hard to construct examples which show that the number 3 in Theorem 1 is best possible. In particular, [1, Example 1] illustrates this.

Finally, it is interesting to note that Theorem 1 fails completely without the boundedness condition on set S , even when S is closed. Consider the following useful example by Hare and Kenelly (which appears in [5 and 3, Example 5]).

EXAMPLE 1. Let $T_n = \{(x, y): n-1 \leq y \leq n, n \leq x+y\}$, and let $S = \bigcup T_n$. Then every finite subset of S is clearly visible via S from a common point, yet S is not starshaped.

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