ON RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS. III

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ABSTRACT. Let G be a finite group, let $H \leq G$, and let π be the set of prime divisors of $|H|$. Assume that whenever two elements of H are G -conjugate then they are H -conjugate. Assume that for all $h \in H^\#$, $(C_G(h) : C_H(h))$ is a π' -number. We prove that H is a π -Hall subgroup and that there exists a normal complement $G_0 = G - (H^\#)^{G, \pi}$. An example shows that the generalization to relative normal complements is not true.

This paper continues the studies made in two earlier papers [2, 3]. Our object here is to prove the following theorems.

THEOREM 1. *Let G be a finite group, let $H \leq G$, and let π be the set of prime divisors of $|H|$. Assume the following conditions:*

- (A) *Whenever two elements of H are G -conjugate then they are H -conjugate.*
- (B₁) *For every nonidentity π -element $h \in H$, $(C_G(h) : C_H(h))$ is a π' -number.*

Then H is a π -Hall subgroup of G , and there exists a unique normal complement $G_0 = G - (H^\#)^{G, \pi}$ of H in G .

Conversely, given a π -Hall subgroup H of G and a normal complement G_0 of H , conditions (A) and (B₁) must hold.

Here we mean that $G_0 \Delta G$, $G_0 H = G$, and $G_0 \cap H = \{1\}$. The statement that H is a π -Hall subgroup means that $|H|$ is divisible only by primes in π and that no prime in π divides $(G : H)$. Each element x of G has a unique decomposition, $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$, into a π -element x_π and a π' -element $x_{\pi'}$. Both x_π and $x_{\pi'}$ are powers of x . Two elements x and y of G belong to the same π -section of G if their π -parts x_π and y_π are G -conjugate. If S is a subset of G we let $S^{G, \pi}$ denote the union of all π -sections of G that intersect S .

Under stronger assumptions than our conditions (A) and (B₁), G. Robinson has proved a theorem asserting the existence of a relative normal complement [4, Theorem D]. His theorem is itself a generalization of earlier results. (See his references.)

An example will show that the generalization to relative normal complements of Theorem 1 is not true. Let G be the p -group described in [2, §5, Example 1]. Let H and H_0 be the subgroups described there. According to [2, Proposition 5.1] we have

THEOREM 2. *Let G be the p -group referred to, with $H_0 \Delta H \leq G$ as described. Then:*

- (i) *Whenever two elements x_1 and x_2 of H are G -conjugate then $x_1 H_0 = x_2 H_0$.*

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(ii) If $x \in H - H_0$ then $C_G(x) = H$.

(iii) There is no relative normal complement G_0 of H over H_0 .

By definition a relative normal complement G_0 of H over H_0 would be a normal subgroup G_0 of G such that $G_0H = G$ and $G_0 \cap H = H_0$.

PROOF OF THEOREM 1. Let P be a Sylow p -subgroup of H for some $p \in \pi$. Suppose P is not a Sylow p -subgroup of G . Then some p -subgroup Q of G contains P as a proper normal subgroup. Then $Z(Q) \cap P \neq \langle 1 \rangle$. Let k be a nonidentity element of $Z(Q) \cap P$. Then $Q \subseteq C_G(k)$ but $Q \not\subseteq H$. Thus $p \mid (C_G(k) : C_H(k))$, contrary to hypothesis (B_1) .

Thus P is a Sylow p -subgroup of G , and H is a π -Hall subgroup of G . Therefore G satisfies the hypotheses of a Theorem of Brauer [1, Theorem 3]. (Also see his Remark 1, p. 81.)

The converse part of our Theorem 1 follows directly from Brauer's Theorem 3 and his Remark 1 [1].

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