SOME EXAMPLES OF CYCLIC VECTORS IN THE DIRICHLET SPACE

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ABSTRACT. We consider the Hilbert space of analytic functions in the open unit disc that have a finite Dirichlet integral. For E , a closed subset of the unit circle with logarithmic capacity zero, we construct a function in this space which is uniformly continuous, vanishes on E , and is cyclic with respect to the shift operator.

For Δ , the open unit disk in the complex plane, let D be the space of functions f analytic on Δ with finite *Dirichlet integral*

$$\iint_{\Delta} |f'|^2 dx dy < \infty.$$

If f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

then

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2$$

defines a norm on D which makes D a Hilbert space. The problem of identifying the *cyclic vectors* in D , that is, those functions f such that the set $\mathcal{P}f = \{pf: p \text{ a polynomial}\}$ is dense in D was studied in [1]. Denote by $[f]$ the closure of $\mathcal{P}f$. The following result was established.

THEOREM A [1, THEOREM 5]. *If f is any function in D whose radial limit vanishes on a set of positive logarithmic capacity then f is not cyclic.*

Recall that if $f \in D$ and T is the unit circle then

$$f^*(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$$

exists for $e^{i\theta} \in T \setminus E$ where E is a set of zero logarithmic capacity depending on f ; see [4, Chapter IV, Théorème I] for a stronger result.

Question 12 of [1] asks if the converse of Theorem A holds for an outer function $f \in D$. That is, letting $Z(f)$ denote the zero set of f^* for an outer function f , if $Z(f^*)$ has zero logarithmic capacity, must f be cyclic? In fact, no examples were given in [1] of cyclic vectors f for which $Z(f)$ was uncountable.

In this note, we modify a construction presented by Carleson (see the proof of Theorem 4 in [2]) to obtain the following result.

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THEOREM B. *Let E be a closed subset of the circle T of zero logarithmic capacity. Then there exists a function $f \in D$ which is continuous on the closed disk $\overline{\Delta}$, cyclic in D and $Z(f) = E$.*

In the literature there are several definitions of logarithmic capacity. We choose to use the one presented in Chapter III of [3]. We prefer the kernel to be positive for $r = |z - w|$, $z, w \in \overline{\Delta}$, and set, following the notation of [3],

$$H(t) = \begin{cases} t + \log 4, & t \geq -\log 4, \\ 0, & t < -\log 4, \end{cases}$$

and

$$K(r) = H(\log 1/r) = \log^+(4/r).$$

It is a simple matter to verify that the sets of capacity zero obtained by using the kernel K as in [3] are the same as the sets of logarithmic capacity zero obtained by other authors.

If E is a bounded Borel set, let Γ_E be the class of positive measures μ with support contained in E such that $U_\mu(z) \leq 1$, $z \in E$, where U_μ is the *logarithmic potential*

$$U_\mu(z) = \int K(|z - w|) d\mu(w).$$

Also, define the *energy integral* $I(\mu)$,

$$I(\mu) = \int U_\mu(z) d\mu(z) = \iint K(|z - w|) d\mu(w) d\mu(z).$$

The (logarithmic) capacity $C_K(E)$ of E is defined by the relation

$$\text{cap}(E) = C_K(E) = \sup_{\mu \in \Gamma_E} \mu(E) = \sup_{\mu \in \Gamma_E} \|\mu\|.$$

If E is compact there exists a unique measure σ in Γ_E such that $\|\sigma\| = \text{cap}(E)$; see Theorem 6, p. 20 in [3]. The potential U_σ is called the *conductor potential* for E .

With this notation and theory established we are ready to begin the proof of Theorem B.

If E is a compact subset of T and $\text{cap}(E) = 0$ we may choose closed subsets I_n of T each consisting of finitely many closed arcs such that

- (i) $E \subseteq I_n$, $n = 1, 2, \dots$,
- (ii) $\sum_{n=1}^{\infty} \text{cap}(I_n)^{1/2} < \infty$,
- (iii) $\bigcap_{n=1}^{\infty} I_n = E$.

Let U_n be the conductor potential for I_n and σ_n the unique measure defining this potential. We claim that

(1) $U_n = 1$ p.p. on I_n , i.e., $U_n = 1$ on I_n except possibly on a set of capacity zero;

(2) $0 < U_n \leq 1$;

(3) $\|\sigma_n\| = \text{cap}(I_n)$.

These three claims follow easily from [3, Theorem 4, p. 19 and Theorem 1, p. 15] and the definition of σ_n . We also claim that

(4) $\hat{\sigma}_n(0)^2 \log^2 4 + \sum_{k=1}^{\infty} |\hat{\sigma}_n(k)|^2/k \leq \text{cap}(I_n)$, where

$$\hat{\sigma}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} d\sigma_n(\theta).$$

For this, from (2) and (3) we have

$$I(\sigma_n) \leq \int U_n(z) d\sigma_n(z) \leq \sigma_n(I_n) = \|\sigma_n\| = \text{cap}(I_n).$$

By [4, Proposition 3, p. 35] we have

$$I(\sigma_n) = \hat{\sigma}_n(0)^2 \log^2 4 + \sum \frac{|\hat{\sigma}_n(k)|^2}{k}.$$

See the proof of [1, Theorem 5] for details. This proves (4).

By representing U_n as the Poisson Integral of $U_n(e^{i\theta})$ and using properties (1) and (2) above, we see that for each n we may choose $0 < r_n < 1$ so close to 1 that

$$U_n(r_n e^{i\theta}) \geq \frac{1}{2}, \quad \text{for } e^{i\theta} \in E;$$

such an r_n exists since I_n is a union of finitely many arcs. Let U_n^* be the harmonic conjugate of U_n on the disk Δ which vanishes at 0 and set

$$\varphi_n(z) = U_n(r_n z) + iU_n^*(r_n z), \quad z \in \bar{\Delta}.$$

Since φ_n is analytic on $\bar{\Delta}$ it is clear that $\varphi_n \in D$. We proceed to estimate $\|\varphi_n\|$. First,

$$\hat{\varphi}_n(0) = \sigma_n(T) \ln 4 = \hat{\sigma}_n(0) \ln 4.$$

Next, for $k > 0$ and $h(t) = \ln 4 / |1 - e^{it}|$,

$$\begin{aligned} \hat{\varphi}_n(k) &= 2r_n^k \hat{U}_n(k) = 2r_n^k \widehat{h * \sigma_n}(k) \\ &= 2r_n^k \hat{h}(k) \hat{\sigma}_n(k) = \frac{r_n^k \hat{\sigma}_n(k)}{k}. \end{aligned}$$

Thus,

$$\|\varphi_n\|^2 \leq |\hat{\sigma}_n(0)|^2 \log^2 4 + \sum_{k=1}^{\infty} (k+1) \frac{|\hat{\sigma}_n(k)|^2}{k^2} \leq 2 \text{cap}(I_n),$$

where we have used (4). It follows now that $\sum \varphi_n$ is in D and

$$\left\| \sum \varphi_n \right\| \leq \sqrt{2} \sum \text{cap}(I_n)^{1/2} < \infty.$$

Furthermore, since $0 < \text{Re } \varphi_n(z) < 1$ for $z \in \Delta$, we have that

$$f(z) = \exp \left(- \sum \varphi_n(z) \right)$$

is in H^∞ ; in fact, $\|f\|_\infty \leq 1$.

We claim that

- (5) $f \in D$;
- (6) $[f] = D$, i.e. f is cyclic in D ;
- (7) f is continuous on $\bar{\Delta}$;
- (8) $Z(f) = E$;

and thus Theorem B is proved.

To prove (5) observe that $f'(z) = f(z) \sum \varphi'_n(z)$. Since $f \in H^\infty$ and $\sum \varphi_n \in D$ it follows easily that $f \in D$.

To verify (6) let $F_N(z) = \exp(-\sum_{n>N} \varphi_n(z))$. Since $\exp(\varphi_n)$ is analytic on a neighborhood of $\bar{\Delta}$ one shows without difficulty that

$$F_N = \prod_{n=1}^N \exp(\varphi_n) \cdot f \in [f].$$

It is easily checked that $\lim_{N \rightarrow \infty} \|F_N - 1\| = 0$. Thus $1 \in [f]$ and it follows that f is cyclic.

To check (7) and (8) we show first that

$$v(z) = \sum_{n=1}^{\infty} \operatorname{Re}(\varphi_n(z)) = \sum_{n=1}^{\infty} U_n(r_n z)$$

is continuous in the extended sense on $\bar{\Delta}$. If $z_0 \in \bar{\Delta}$ and is a positive distance from E , i.e. $\operatorname{dist}[z_0, E] > 0$, then it follows that for n sufficiently large

$$\operatorname{dist}[r_n z, I_n] \geq \delta > 0$$

for all points z in some open ball centered at z_0 . Thus,

$$U_n(z) = \int_{I_n} \log \frac{4}{|r_n z - e^{i\theta}|} d\sigma_n(\theta) \leq \left(\log \frac{4}{\delta} \right) \|\sigma_n\| = \left(\log \frac{4}{\delta} \right) \operatorname{cap}(I_n).$$

The Weierstrass M -test implies that v is continuous in a neighborhood of z_0 .

Next, for $z_0 \in E$ it follows from the choice of r_n that $v(z_0) = \infty$. Since $\operatorname{Re} \varphi_n$ is continuous at z_0 and $\operatorname{Re} \varphi_n(z_0) \geq \frac{1}{2}$, there is a ball $B(z_0, \varepsilon)$ centered at z_0 of radius ε such that $z \in \bar{\Delta} \cap B(z_0, \varepsilon)$ implies $\operatorname{Re} \varphi_n(z) \geq \frac{1}{4}$ for $n = 1, 2, \dots, N$, where N has been chosen beforehand. Thus $v(z) \geq N/4$ for $|z - z_0| < \varepsilon$. This shows that

$$\lim_{z \rightarrow z_0} v(z) = \infty, \quad z \in \bar{\Delta},$$

that is, v is continuous in the extended sense.

The calculations above show that $|f| = e^{-v}$ is continuous on $\bar{\Delta}$ and that $Z(f) = E$. To show f is continuous on $\bar{\Delta}$ we need therefore only check continuity at $e^{it} \in T \setminus E$. Let γ_t be the line segment from 0 to $r_n e^{it}$ and let $\partial/\partial n$ denote the outer normal derivative. Then

$$\begin{aligned} U_n^*(r_n e^{it}) &= \int_{\gamma_t} \frac{\partial U_n(z)}{\partial n} |dz| \\ &= \int_{\gamma_t} \left[\frac{\partial}{\partial n} \int_{I_n} \log \frac{4}{|z - e^{i\theta}|} d\sigma_n(\theta) \right] |dz|, \end{aligned}$$

and

$$|U_n^*(r_n e^{it})| \leq \int_{\gamma_t} \int_{I_n} \frac{c}{|z - e^{i\theta}|} d\sigma_n(\theta) |dz| \leq \frac{2c\|\sigma_n\|}{\operatorname{dist}[e^{it}, E]}$$

for n sufficiently large. Since $\sum \|\sigma_n\| < \infty$ it follows that $\sum U_n^*$ is continuous at $e^{it} \in T \setminus E$ and the proof is complete.

We wish to make two remarks:

(1) A modification of the above proof of cyclicity can yield that Carleson's original examples (see [2, Theorem 4]) are cyclic in D . Of course, these examples may not necessarily be continuous in $\bar{\Delta}$.

(2) If D_α , $0 < \alpha < 1$, is the Hilbert space of analytic functions for which $\|f\|^2 = \sum_{n=0}^{\infty} (n+1)^\alpha |\hat{f}(n)|^2$ is finite, then by using a suitable Bessel capacity instead of logarithmic capacity, one can prove theorems for D_α corresponding to [1, Theorem 5] and Theorem B of this note.

REFERENCES

1. L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. **285** (1984), 269–304.
2. L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. **87** (1952), 325–345.
3. ———, *Selected problems on exceptional sets*, Van Nostrand, Princeton, N.J., 1967.
4. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Actualités Sci. Indust., no. 1301, Hermann, Paris, 1963.

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