SHORTER NOTES

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ON A PROBLEM OF BERMAN CONCERNING RADIAL LIMITS

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ABSTRACT. Given a G_δ -subset E of the unit circle T such that E is of measure zero, we prove that there exists a nonvanishing function $g \in H^\infty$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E . This answers a problem of R. Berman (Proc. Amer. Math. Soc. **92** (1984), 64–66).

Let $U = \{z: |z| < 1\}$ and $T = \{z: |z| = 1\}$. Recently, R. Berman [1, Remark (1), p. 66] asked the following question: If E is a G_δ -subset of T such that E is of measure zero, does there exist a nonvanishing function $g \in H^\infty(U)$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E ? Previously, R. Cahill [2, Theorem 5, p. 171] proved that, under the same conditions on E , there exists a nonvanishing function $g \in H^\infty(U)$ such that the modulus of g has a radial limit at each point of T and this radial limit is zero at each point of E . In this note, we will give an affirmative answer to Berman's question by showing that Cahill's function g has a radial limit at each point of T .

THEOREM. *If E is a G_δ -subset of T such that E is of measure zero, then there exists a nonvanishing function $g \in H^\infty(U)$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E .*

PROOF. Z. Zahorski [5] proved that there exists a monotone increasing differentiable function ϕ on $(-\pi, \pi)$ such that $\phi'(\theta) = Z(\theta)$ exists everywhere on $(-\pi, \pi)$ and $Z(\theta) = \infty$ if and only if $e^{i\theta} \in E$. In [2], Cahill extended $Z(\theta)$ to the whole real line so that $Z(\theta)$ is periodic with period 2π , and defined a function

$$h(z) = u(z) + iv(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} Z(\theta) d\theta$$

such that $u(z) > 0$ for each $z \in U$ and $u(z)$ has a radial limit at each point of T , where this radial limit is $+\infty$ at each point of E . For our purposes, the important point in Cahill's reasoning is that

$$(1) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h (Z(\theta + t) - Z(\theta - t)) dt = 0 \quad \text{for each } \theta \in [-\pi, \pi].$$

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(This was proved by Zahorski.) By a theorem due to Plessner [3], we have

$$(2) \quad v(re^{i\theta}) \rightarrow -\frac{1}{2\pi} \int_0^\pi (Z(\theta+t) - Z(\theta-t)) \cot \frac{t}{2} dt \quad \text{as } r \rightarrow 1-$$

for each $\theta \in [-\pi, \pi]$ (see [4, Theorem IV.27, p. 162]). In fact, the argument in [4] shows that (1) implies that the right side of (2) is finite for each $\theta \in [-\pi, \pi]$, so $v(z)$ has a finite radial limit at each point of T . Cahill defined $g(z) = \exp(-h(z)) = \exp(-u(z) - iv(z))$, and thus $g(z)$ has a radial limit at each point of T , since $g(z)$ has the radial limit zero at each point of E , and both $u(z)$ and $v(z)$ have finite radial limits at each other point of T .

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