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THE CARTAN MATRIX AS AN INDICATOR OF FINITE GLOBAL DIMENSION FOR ARTINIAN RINGS

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ABSTRACT. A left serial ring has finite global dimension if and only if its Cartan matrix has determinant equal to 1.

1. Introduction. A slight modification of a counting argument given by Eilenberg (see [**2**, Proposition 21; **13**, Proposition 1.1]) three decades ago shows that the determinant of the Cartan matrix C of a left artinian ring R of finite global dimension is always 1 or -1. No instances are known in which the value -1 is attained. In a recent paper, Zacharia [**14**] proved that the determinant is, in fact, 1 whenever $\operatorname{gl\,dim} R \leqslant 2$. We continue this line by establishing $\det C = 1$ for left serial rings of finite global dimension. What is more surprising is the fact that, conversely, $\det C = 1$ guarantees finite global dimension for any left serial ring. This provides a simple finitary procedure for determining whether a left serial ring is of finite dimension. The latter implication does not extend to arbitrary left artinian rings; we exhibit an example which deviates "as slightly as possible" from being left serial, but which nevertheless combines $\det C = 1$ with infinite global dimension.

The phenomenon just described already occurs for artinian rings of Loewy length 2. On the other hand, for Loewy length 2, the Cartan matrix as a whole still reflects whether or not the global dimension is finite; this was observed by Jans and Nakayama [**8**, Proposition 10]. It is natural to look at Loewy length 3 next. Here, knowledge of the Cartan matrix no longer suffices for the distinction between finite and infinite global dimension; in fact, we construct two artinian rings of Loewy length 3 with identical Cartan matrices, one of which has finite, the other infinite, global dimension.

It is still open whether finite global dimension implies $\det C = 1$ for all left artinian rings. The positive answers in the special cases $\operatorname{gl\,dim} R \leqslant 2$, R left serial, R of Loewy length $\leqslant 2$ are based on the existence of a simple left module of projective dimension $\leqslant 1$ (discarding the corresponding primitive idempotent gives rise to a finite induction). Our final disillusioning example demonstrates that this road will not lead to a positive decision in the general setting.

Throughout, R will denote a left artinian ring with Jacobson radical J, and $\{e_1, \ldots, e_n\}$ will be a complete orthogonal set of pairwise nonisomorphic primitive idempotents. In particular, the isomorphism types of the simple modules (module

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stands for left module unless otherwise indicated) are $S_1, \ldots, S_n$ where $S_i = Re_i/Je_i$. Recall that the Cartan matrix $C = C({}_RR)$ is the $n \times n$ matrix with integral entries c_{ij} which are defined as follows: c_{ij} is the number of copies of the simple module S_i appearing in a composition series for Re_j. Clearly, the ring R has the same Cartan matrix as its basic ring. Moreover, if $R = R_1 \times R_2$ is a ring decomposition, C is a block diagonal matrix with blocks $C(R_1)$ and $C(R_2)$. Hence, in the sequel we may always assume that R is *basic* and *indecomposable*.

2. Left serial rings. Recall that a left artinian ring R is called *left serial* if each of the indecomposable projective modules Re_i is uniserial, that is, contains a unique composition series. *Serial* rings are rings satisfying this condition on both sides.

Examples of serial rings can, for instance, be found in [**4**] and [**11**]; for examples of left serial rings which are not serial, see [**9**].

The sequence $Re_1, \ldots, Re_n$ is called a (left) Kupisch series if Re_i is a projective cover of Je_{i+1} for $1 \leqslant i \leqslant n-1$, and Re_n is a projective cover for Je_1 or $Je_1 = 0$. By [**10**] such series exist whenever R is serial; if R is QF also, then all Re_i have the same composition length [**11**, Theorem 19]. Thus, the following computation yields, in particular, the Cartan matrices of QF serial rings.

LEMMA 1. *Let R be left serial, and suppose that R has a Kupisch series $Re_1, \ldots, Re_n$, all of whose members have the same composition length m. Write $m = an + r$ with $0 \leqslant r < n$. Then C has the following form*:

$$c_{ij} = \begin{cases} a+1 & \text{if } 0 \leqslant j - i < r \text{ or } n - r < i - j \leqslant n, \\ a & \text{otherwise}. \end{cases}$$

(In other words, for $r > 0$, the matrix C has entries $a + 1$ on the main diagonal, on the next $r - 1$ superdiagonals, and on the subdiagonals from $n - r + 2$ down, and the remaining entries are all a.)

PROOF. The sequence of composition factors of Re_j is S_j, $S_{\overline{j-1}}, \ldots,$ $S_{\overline{j-(n-1)}}$, S_j, $S_{\overline{j-1}}, \ldots,$ ($\bar{k}$ stands for the least positive remainder of k modulo n); it continues for m terms. Thus, there are $a + 1$ copies of the first r candidates in the list and a copies of the others. □

The matrices described in Lemma 1 are types of circulant matrices. We record a special case of [**1**, Problem 27, p. 81] as

LEMMA 2. *Let C be a matrix as in the previous lemma. Then*

$$\det C = \begin{cases} m & \text{if } \gcd(m, n) = 1, \\ 0 & \text{if } \gcd(m, n) \neq 1. \end{cases}$$

LEMMA 3. *If R is left serial with a simple module of finite projective dimension, then R has a simple module of projective dimension $\leqslant 1$.*

PROOF. Either R has a simple projective module, or the minimal projective resolution of a simple module of finite projective dimension $\geqslant 1$ provides us with a proper monomorphism $O \to Re_i \to Re_j$ whose image is $J^m e_j$, say. In the latter case, if $Re_k \xrightarrow{g} J^{m-1}e_j \to 0$ is a projective cover, g induces a split epimorphism $Je_k \to J^m e_j \cong Re_i$. But Je_k is indecomposable, and hence this is an isomorphism. It follows that Re_k/Je_k has projective dimension 1. □

One direction of the equivalence in the next lemma is due to Zacharia [**14**, Lemma 2]; the inequality appears in Gustafson [7] with the added hypothesis that R be serial and $\operatorname{gl\,dim} R < \infty$. In establishing the bridge between Cartan matrix and global dimension, we will use it to successively reduce the number of primitive idempotents.

LEMMA 4. *Let R be left serial and $e = 1 - e_1$. Provided that* $\operatorname{p\,dim} S_1 \leqslant 1$, *the left global dimension of R is finite if and only if the left global dimension of eRe is finite, and* $\operatorname{l\,gl\,dim} R \leqslant \operatorname{l\,gl\,dim} eRe + 2$.

PROOF. First suppose that $\operatorname{l\,gl\,dim} eRe$ is finite.

If $\operatorname{p\,dim} S_1 = 0$, we have $Je_1 = 0$; if $\operatorname{p\,dim} S_1 = 1$, then $Je_1/J^2e_1 \not\cong S_1$. In both cases we find $\operatorname{Ext}^1(S_1, S_1) = 0$.

Now let $i \neq 1$ and $\operatorname{p\,dim}_{eRe} eS_i = m$. In order to verify that $\operatorname{p\,dim} S_i \leqslant m + 2$, consider a projective resolution

$$\cdots \to P_k \xrightarrow{f_k} P_{k-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} S_i \to 0$$

of S_i, where all the P_k are indecomposable. As Zacharia [**14**, p. 355] observed (note that his argument for $\operatorname{p\,dim} S_1 = 1$ works equally well for $\operatorname{p\,dim} S_1 = 0$), the sequence

$$\cdots \to eP_k \to eP_{k-1} \to \cdots \to eP_1 \to eP_0 \to eS_i \to 0$$

is then an eRe-projective resolution of the eRe-module eS_i. In particular, $f_m(eP_m)$ is projective and nonzero. Since eP_m is indecomposable, we infer that $f_m|_{eP_m}$ is monic. Set $T = \operatorname{Soc} P_m$.

If $T \not\cong S_1$, we have $0 \neq f_m(eT) \subset f_m(T)$, whence $f_m\colon P_m \to P_{m-1}$ is also monic, and $\operatorname{p\,dim} S_i \leqslant m$.

If $T \cong S_1$, we invoke $\operatorname{Ext}^1(S_1, S_1) = 0$ to obtain a submodule L of P_m, namely $L = \operatorname{Soc}^2 P_m$, such that $L/T \not\cong S_1$. From $0 \neq f_m(eL) \subseteq f_m(L)$, it follows that $L \not\subseteq \ker f_m$ and, consequently, $\ker f_m \subseteq L$. From $\ker f_m \neq L$ we conclude further that $\ker f_m$ equals either T or 0. In the latter case we have $\operatorname{p\,dim} S_i \leqslant m$ as above. In the former we obtain an exact sequence

$$0 \to Je_1 \to Re_1 \to P_m \xrightarrow{f_m} \cdots \to P_0 \to S_i \to 0,$$

which, in view of the projectivity of Je_1, yields $\operatorname{p\,dim} S_i \leqslant m + 2$.

Conversely, finiteness of $\operatorname{l\,gl\,dim} R$ implies finiteness of $\operatorname{l\,gl\,dim} eRe$, since, as we have already remarked, multiplication of an R-projective resolution of a simple S_i ($i \geqslant 2$) by e results in an eRe-projective resolution of eS_i. □

The last of our preparatory lemmas is due to Zacharia [**14**, Theorem B]. We include a particularly simple argument.

LEMMA 5. *Suppose that the projective dimension of $S_1 = Re_1/Je_1$ is $\leqslant 1$ and set $e = 1 - e_1$. Then* $\det C(eRe) = \det C(R)$.

PROOF. By hypothesis,

$$Je_1 = (Re_2)^{m_2} \oplus \cdots \oplus (Re_n)^{m_n} \quad \text{with } m_i \geqslant 0.$$

Denoting the jth column of the Cartan matrix C of R by

$$c_j = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{pmatrix},$$

we obtain

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{j=2}^{n} m_j c_j.$$

Thus, subtraction of m_j times the jth column from the first column for $j = 2, \dots, n$ yields the matrix

$$\begin{pmatrix} 1 & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

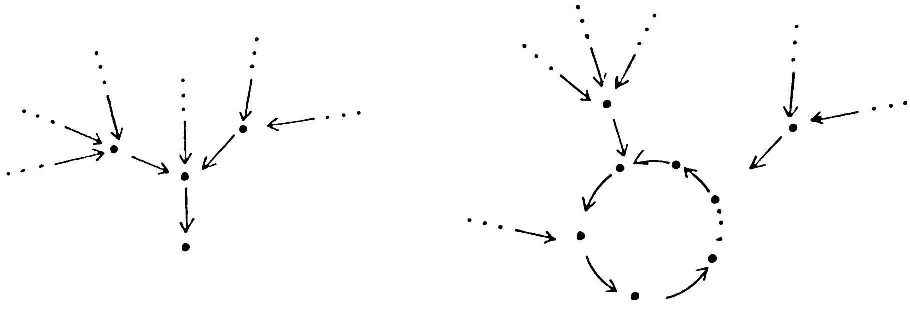
and consequently,

$$\det C = \det \begin{pmatrix} c_{22} & \cdots & c_{2n} \\ \vdots & & \vdots \\ c_{n2} & \cdots & c_{nn} \end{pmatrix}.$$

But the latter matrix is the Cartan matrix of eRe. □

Recall that the quiver of a left artinian ring R is a directed graph with vertices $e_1, \dots, e_n$ and precisely r arrows $e_i \to e_j$ if Je_i/J^2e_i contains r copies of S_j (see, for example, [**6**, pp. 88, 119]). Since our ring is indecomposable, its quiver is connected as an undirected graph.

In the case of a left serial ring, the quiver has a particularly tractable form: it is either a rooted tree or a graph obtained from a rooted tree by replacing its root by an oriented cycle. (Here we go against the usual convention by having the arrows point towards the root).



To see this, we first note that R is left serial if and only if at most one arrow emanates from any given vertex. As in [5], a quasi-order can be introduced on the set of vertices via $e_i < e_j$ if $i = j$ or if there is an oriented path from e_j to e_i. Let e_1 be a minimal element with respect to this quasi-order, and consider the set $S = \{e_i | e_i \leqslant e_1\}$; together with the arrows between these vertices, S either forms an oriented cycle or a loop ⟲, or else a graph consisting just of one point ●. Since there are never two arrows leaving a given vertex (and since the underlying undirected graph is connected), any vertex belonging to a cycle or a loop lies in S.

Now weight each vertex e_i by the composition length c_i of Re_i. Then the weights along an oriented path in the quiver of a left serial ring R behave like the admissible sequences of serial rings [4]. More precisely, if $e_i \to e_j$ is an arrow in the quiver, that is, Re_j is a projective cover of Je_i, then clearly $c_j \geqslant c_i - 1$, and $c_j = c_i - 1$ if and only if $\operatorname{p dim} Re_i/Je_i = 1$.

The following remark evokes the situation described in Lemma 1. If $e_1 = e_{i_1} \to e_{i_2} \to \cdots \to e_{i_k} \to e_1$ is an oriented cycle, then $Re_{i_{m+1}}$ is a projective cover of Je_{i_m} and Re_{i_1} is a projective cover of Je_{i_k}; in particular, the only simple factors appearing in the composition series for the Re_{i_m} are $S_{i_1}, S_{i_2}, \dots, S_{i_k}$. Moreover, absence of candidates with projective dimension $\leqslant 1$ among the S_{i_m} forces all the composition lengths c_{i_m} to be equal.

Theorem 6. *Given a left serial ring R, the determinant of its Cartan matrix is* 1 *if and only if its left global dimension is finite. In any case the determinant is nonnegative.*

Proof. We revert to the usual restrictions on R. If $\operatorname{l gl dim} R < \infty$, then the combination of Lemmas 3–5 allows successive elimination of primitive idempotents corresponding to simple modules of projective dimension $\leqslant 1$ until we are left with one idempotent. But, in this situation, finite global dimension is tantamount to semisimplicity, whence $\det C = 1$.

Now assume that $\operatorname{l gl dim} R = \infty$. By induction on the number n of primitive idempotents we will show that either $\det C = 0$ or $\det C > 1$. The case $n = 1$ is trivial. For the induction step we may start with $n > 1$ primitive idempotents e_i, none of which gives rise to a simple module Re_i/Je_i of projective dimension $\leqslant 1$ (otherwise Lemmas 4 and 5 would permit us to discard one idempotent and invoke the induction hypothesis). That the quiver of R is a tree is impossible, since its root would correspond to a projective module of length 1. Therefore the quiver contains a cycle or loop with vertices $e_1, \dots, e_k$ ($k \geqslant 1$), say. Since the composition series of $Re_1, \dots, Re_k$ involve only the simple modules $S_1, \dots, S_k$, the Cartan matrix of R has block form

$$\begin{bmatrix} C_1 & X \\ 0 & C_2 \end{bmatrix},$$

where C_1 is a $k \times k$ matrix as treated in Lemmas 1 and 2, whence $\det C_1 = 0$ or $\det C_1 > 1$. Note that C_2 is the Cartan matrix of eRe, where $e = 1 - e_1 - \cdots - e_k$. If $\operatorname{l gl dim} eRe < \infty$, we have $\det C_2 = 1$ by the first part of the proof otherwise the induction hypothesis yields $\det C_2 \geqslant 0$. In either case, $\det C = (\det C_1)(\det C_2)$ is either zero or greater than 1. □

COROLLARY 7. *Let R be an Artin algebra which is left or right serial, and let C be its left Cartan matrix. Then* $\det C = 1$ *if and only if* $\operatorname{gl\,dim} R < \infty$. *In any case* $\det C \geqslant 0$.

PROOF. It suffices to note that in the case of an Artin algebra the left and right Cartan matrices have the same determinant. This follows from a straightforward modification of an argument due to Nakayama [**12**, Theorem 3]. □

3. Negative examples and positive remarks. For any left artinian ring, finite left global dimension narrows the range of possible values for $\det C$ to ± 1 (see [**2**, Proposition 21; **13**, Proposition 1.1]).

Our first two examples demonstrate that both $+1$ and -1 can also occur as Cartan determinants of artinian rings of *infinite* left global dimension. The examples we exhibit are both algebras over a field F of Loewy length 2 and F-dimension 5 (one can easily convince oneself that F-dimension 5 is the lowest possible for either of these phenomena to occur). An example featuring $\det C = -1$ was already given by Eilenberg, Ikeda and Nakayama in [**3**]; however, their example is of dimension 12 over the base field.

EXAMPLE 8. *Combining infinite global dimension with* $\det C = 1$. Let F be any field and R the subring of $M_4(F)$ consisting of all matrices of the form

$$\begin{bmatrix} a & u & 0 & v \\ 0 & b & 0 & 0 \\ 0 & 0 & b & w \\ 0 & 0 & 0 & a \end{bmatrix}, \quad \text{where } a, b, u, v, w \in F.$$

R has two primitive idempotents: $e_1 = e_{11} + e_{44}$ and $e_2 = e_{22} + e_{33}$. It is straightforward to check $J^2 = 0$, $e_1Je_1 \neq 0$, $e_2Je_1 \neq 0$, $e_1Je_2 \neq 0$ and $e_2Je_2 = 0$, whence $Je_1 \cong S_1 \oplus S_2$ and $Je_2 \cong S_1$ (again, we set $S_i = Re_i/Je_i$). It follows that $\operatorname{gl\,dim} R = \infty$, whereas $C = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ □.

EXAMPLE 9. *Combining infinite global dimension with* $\det C = -1$. Starting again with a field F, this time we let R be the subring of $M_5(F)$ consisting of all matrices of the form

$$\begin{bmatrix} a & 0 & u & v & 0 \\ 0 & b & 0 & 0 & w \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}, \quad \text{where } a, b, u, v, w \in F.$$

Putting $e_1 = e_{11} + e_{55}$ and $e_2 = e_{22} + e_{44}$, we obtain $Je_1 \cong S_2$ and $Je_2 \cong S_1 \oplus S_1$, whence $\operatorname{gl\,dim} R = \infty$ and $C = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]$. □

Even though, for general artinian rings of Loewy length 2, the *determinant* of the Cartan matrix fails to distinguish between finite and infinite global dimension, the Cartan matrix as a whole still does. In fact, from a result of Jans and Nakayama [**8**, Proposition 10], it follows that, given a left artinian ring R of Loewy length 2,

$$\operatorname{l\,gl\,dim} R \leqslant k \Leftrightarrow (C - I)^{k+1} = 0$$

and

$$\operatorname{l\,gl\,dim} R < \infty \Leftrightarrow (C - I)^{n} = 0;$$

here I denotes the $n \times n$ identity matrix. We include a very short and elementary argument for a mild reinforcement of these equivalences.

REMARK 10. *If* $J^2 = 0$, *the following statements are equivalent*:

(a) $\operatorname{l\,gl\,dim} R < \infty$;

(b) C *is an upper triangular matrix with entries* 1 *along the main diagonal* (*for a suitable arrangement of the primitive idempotents*).

In any case, the projective dimensions of the simple modules can be read off the columns of the Cartan matrix as follows: The columns containing a 1 in position (j, j) and 0 elsewhere correspond to the simple projectives Re_j; say this occurs for the columns with indices j in D_0. The columns of index $j \notin D_0$ containing a 1 in position (j, j) and additional nonzero entries only in the positions (l, j), with $l \in D_0$, correspond to the projectives Re_j such that $\operatorname{p\,dim} S_j = 1$; say this is the case for the columns with indices j in D_1. For, columns with indices $j \notin D_0 \cup D_1$ containing a 1 in position (j, j) and further nonzero entries only in the positions (l, j), with $l \in D_0 \cup D_1$, correspond to the projectives Re_j such that $\operatorname{p\,dim} S_j = 2$. When this procedure of successively filtering out the simple modules of projective dimension $0, 1, 2, 3, \ldots$ comes to a halt, the leftover columns correspond to the projectives Re_j such that $\operatorname{p\,dim} S_j = \infty$.

PROOF. $J^2 = 0$ implies $Je_j = S_1^{t_{j1}} \oplus \cdots \oplus S_n^{t_{jn}}$ with $t_{jn} \geqslant 0$. Thus, $\operatorname{p\,dim} S_j = k < \infty$ is tantamount to $t_{jj} = 0$ and $\sup\{\operatorname{p\,dim} S_l \colon t_{jl} \neq 0\} = k - 1$. If we arrange the primitive idempotents such that $\operatorname{p\,dim} S_j \leqslant \operatorname{p\,dim} S_{j+1}$, the claimed equivalence follows immediately, and so does the supplementary statement. □

Our final example settles two issues with one stroke. On the one hand, it shows that, among the left artinian rings of Loewy length 3, knowledge of the Cartan matrix no longer suffices to determine whether the left global dimension is finite. On the other hand, the first of the rings we construct has finite global dimension, but does not have any simple modules of projective dimension $\leqslant 1$. This contrasts with the rings we studied in §2, with the case $J^2 = 0$, and with the rings investigated by Zacharia [**14**].

EXAMPLE 11. Let F be any field.

(a) Let R be the subring of $M_7(F)$ consisting of all matrices of the form

$$\begin{bmatrix} c & 0 & x & u & w & m & q \\ 0 & b & y & 0 & v & 0 & r \\ 0 & 0 & a & 0 & z & 0 & s \\ 0 & 0 & 0 & b & 0 & n & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}.$$

There are three primitive idempotents; $e_1 = e_{33}$, $e_2 = e_{22} + e_{44} + e_{55}$, $e_3 = e_{11} + e_{66} + e_{77}$, and the corresponding projective left modules Re_j have the following structure: $Je_1 \cong S_2 \oplus S_3$, $Je_2 \cong Re_1 \oplus S_3$, $Je_3 = Re_1 \oplus L$ with $L/JL \cong S_2$ and $JL \cong S_3$. From this information we derive projective resolutions of the simple modules

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Re_1 & & \longrightarrow & & Re_1 \oplus Re_2 & & \longrightarrow & & Re_3 & \longrightarrow & S_3 & \longrightarrow & 0 \\
 & & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & & \\
 & & & & Re_1 & & & & Je_3 & & & & & &
\end{array}$$

$(\operatorname{p\,dim} S_3 = 2)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Je_3 & \longrightarrow & Re_1 \oplus Re_3 & & \longrightarrow & & Re_2 & \longrightarrow & S_2 \longrightarrow 0 \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & Je_2 & & & &
\end{array}$$

$(\operatorname{p\,dim} S_2 = 3)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Je_2 \oplus Je_3 & \longrightarrow & Re_2 \oplus Re_3 & & \longrightarrow & & Re_1 & \longrightarrow & S_1 \longrightarrow 0 \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & Je_1 & & & &
\end{array}$$

$(\operatorname{p\,dim} S_1 = 4)$.

In particular, we see that $\operatorname{gl\,dim} R = 4$. Moreover, counting composition factors yields the Cartan matrix

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(b) Modifying R by moving the entry n from position $(4, 6)$ to position $(2, 6)$, we arrive at a ring $\tilde{R}$ with $\operatorname{gl\,dim} \tilde{R} = \infty$ (since $\tilde{S}_3$ is isomorphic to a direct summand of $\tilde{J}e_3$) that has the same Cartan matrix as R. □

REFERENCES

1. P. J. Davis, *Circulant matrices*, Wiley, New York, 1979.

2. S. Eilenberg, *Algebras of cohomologically finite dimension*, Comment. Math. Helv. **28** (1954), 310–319.

3. S. Eilenberg, M. Ikeda and T. Nakayama, *On the dimension of modules and algebras*. I, Nagoya Math. J. **8** (1955), 49–57.

4. K. R. Fuller, *Generalized uniserial rings and their Kupisch series*, Math. Z. **106** (1968), 248–260.

5. K. R. Fuller and J. Haack, *Rings with quivers that are trees*, Pacific J. Math. **76** (1978), 371–379.

6. R. Gordon and E. L. Green, *Modules with cores and amalgamation of indecomposable modules*, Mem. Amer. Math. Soc. **187** (1977).

7. W. H. Gustafson, *Global dimension in serial rings*, 1983 (typescript).

8. J. P. Jans and T. Nakayama, *On the dimension of modules and algebras*. VII, Nagoya Math. J. **11** (1957), 67–76.

9. G. J. Janusz, *Some left serial algebras of finite type*, J. Algebra **23** (1972), 404–411.

10. H. Kupisch, *Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung*, J. Reine Angew. Math. **201** (1959), 100–112.

11. I. Murase, *On the structure of generalized uniserial rings*. I, II, III, Sci. Papers College Gen. Ed. Univ. Tokyo **13** (1963), 1–22; ibid. **13** (1963), 131–158; ibid. **14** (1964), 11–25.

12. T. Nakayama, *Some studies on regular representations, induced representations and modular representations*, Ann. of Math. (2) **39** (1938), 361–369.

13. G. V. Wilson, *The Cartan map on categories of graded modules*, J. Algebra **85** (1983), 390–398.

14. D. Zacharia, *On the Cartan matrix of an Artin algebra of global dimension two*, J. Algebra **82** (1983), 353–357.

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