

THE AUTOMORPHISM TOWER PROBLEM

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ABSTRACT. It is shown that the automorphism tower of an infinite centerless group terminates. For each ordinal α , a group is constructed whose automorphism tower terminates in exactly α steps.

Let G be a centerless group. Then the automorphism tower of G is defined inductively by

$$\begin{aligned} G_0 &= G, \\ G_\alpha &\hookrightarrow \text{Aut } G_\alpha = G_{\alpha+1}, \quad \text{via the natural map,} \\ G_\lambda &= \bigcup_{\alpha < \lambda} G_\alpha \quad \text{when } \lambda \text{ is a limit ordinal.} \end{aligned}$$

The automorphism tower is said to terminate if there is an ordinal α such that $G_\alpha = G_\beta$ for all $\beta > \alpha$. A classical result of Wielandt [7] says that if G is finite then the automorphism tower terminates after finitely many steps. Rae and Roseblade [3] showed that this is also true if G is a Černikov group. If G is a polycyclic group, then Hulse [2] has proved that the automorphism tower terminates in at most countably many steps. Whether the automorphism tower of an arbitrary centerless group terminates has been an open question for some time. This is resolved by the following theorem:

THEOREM 1. *If G is an infinite centerless group, then the automorphism tower of G terminates in at most $(2^{|G|})^+$ steps.*

Throughout this paper cardinals are taken to be initial ordinals. If κ is a cardinal, then κ^+ denotes the next greatest cardinal. As usual, $|X|$ is the cardinality of the set X .

The proof of Theorem 1 makes use of the fact that each successor cardinal κ^+ is a regular cardinal. This means that if X is a set of cardinality κ^+ then X cannot be expressed as a union of $\theta < \kappa^+$ sets, $X = \bigcup_{\alpha < \theta} X_\alpha$, with $|X_\alpha| < \kappa^+$ for all $\alpha < \theta$.

PROOF OF THEOREM 1. By Lemma 8.1.1 of Hulse [2], $C_{G_\alpha}(G) = 1$ for all ordinals α . If $\varphi \in N_{G_\alpha}(G)$, then $\varphi \upharpoonright G = \varphi_1 \in G_1$, so $\varphi\varphi_1^{-1} \in C_{G_\alpha}(G) = 1$ and $\varphi = \varphi_1$. It follows that $N_{G_\alpha}(G) = G_1$ for all $\alpha \geq 1$. Thus, $[G_{\alpha+1} : G_1]$ equals the cardinality of the set of all conjugates of G in $G_{\alpha+1}$. Each of the latter is contained in G_α ; thus,

$$[G_{\alpha+1} : G_1] \leq |G_\alpha|^{|G|}$$

and

$$|G_{\alpha+1}| \leq |G_1| \cdot |G_\alpha|^{|G|} = |G_\alpha|^{|G|}.$$

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An easy induction shows that

$$|G_\alpha| \leq 2^{|G|}$$

for all $\alpha < \lambda = (2^{|G|})^+$.

Now, assume that the automorphism tower does not terminate in less than λ steps. Then $|G_\lambda| = \lambda$.

Let $\varphi \in \text{Aut } G_\lambda$. Using the fact that λ is a regular cardinal, it is easily shown that there is an $\alpha < \lambda$ such that $G_\alpha^\varphi = G_\alpha$; indeed, the set C of such α is unbounded.

For each $\alpha \in C$ we have $\varphi \upharpoonright G_\alpha \in \text{Aut } G_\alpha = G_{\alpha+1}$ so that $\varphi \upharpoonright G_\alpha = i_{g_\alpha} \upharpoonright G_\alpha$, where i_{g_α} is the inner automorphism of G_λ induced by some $g_\alpha \in G_{\alpha+1}$. If $\alpha, \beta \in C$ then $g_\alpha g_\beta^{-1} \in C_{G_\lambda}(G)$ and $g_\alpha = g_\beta$. Since C is unbounded, it follows that φ is an inner automorphism of G_λ . Therefore, the automorphism tower of G terminates at G_λ . \square

COROLLARY. *An arbitrary infinite group G can be embedded as an ascendant subgroup of a complete group G^* such that $|G^*| \leq (2^{|G|})^+$.*

PROOF. By a standard procedure, embed G subnormally in a centerless group G_1 such that $|G| = |G_1|$. Now, embed G_1 in the terminal group of its automorphism tower. \square

It is not known yet whether $(2^{|G|})^+$ is the best possible bound. The next result shows that the bound does depend on the cardinality of G .

THEOREM 2. *Suppose that α is an ordinal and $\kappa = \max\{\omega, |\alpha|\}$. Then there exists a centerless group G of cardinality κ such that the automorphism tower of G terminates in exactly α steps.*

First, another definition is required. If H is a subgroup of G , then the normalizer tower of H in G is defined inductively by

$$\begin{aligned} N_0(H) &= H, \\ N_{\alpha+1}(H) &= N_G(N_\alpha(H)), \\ N_\lambda(H) &= \bigcup_{\alpha < \lambda} N_\alpha(H), \quad \text{when } \lambda \text{ is a limit ordinal.} \end{aligned}$$

LEMMA. *Let α be an ordinal. There exist groups $H(\alpha) < F(\alpha)$ such that*

- (a) $|F(\alpha)| \leq \max\{\omega, |\alpha|\}$,
- (b) *the normalizer tower of $H(\alpha)$ in $F(\alpha)$ terminates in exactly α steps; further, $N_\alpha(H(\alpha)) = F(\alpha)$.*

PROOF. The groups $H(\alpha)$ and $F(\alpha)$ are constructed inductively. Define

$$H(1) = \text{Alt}(4) < \text{Sym}(4) = F(1).$$

If $\alpha = \beta + 1$, let $F(\alpha) = F(\beta) \text{ wr } C_2$ and take $H(\alpha)$ to be the subgroup $H(\beta) \times 1$ of the base group. If α is a limit ordinal, define

$$H(\alpha) = \text{Dr}_{\beta < \alpha} H(\beta) < \text{Dr}_{\beta < \alpha} F(\beta) = F(\alpha)$$

(Dr denotes the restricted direct product). \square

PROOF OF THEOREM 2. Fried and Kollár [1] have shown that there exists a field K of cardinality κ such that $\text{Aut } K = F(\alpha)$. By Schreier and van der Waerden [6],

$$\text{Aut PSL}(2, K) = \text{PTL}(2, K) = \text{PGL}(2, K) \rtimes F(\alpha).$$

We claim that if $G = \mathrm{PGL}(2, K) \rtimes H(\alpha)$, then for each $\beta \leq \alpha$,

$$G_\beta = \mathrm{PGL}(2, K) \rtimes N_\beta(H(\alpha)),$$

the β th term of the normalizer tower of G in $\mathrm{Aut} \mathrm{PSL}(2, K)$. Suppose that this is true for $\beta < \alpha$ and consider $G_{\beta+1} = \mathrm{Aut} G_\beta$. Since $\mathrm{PSL}(2, K)$ is the unique minimal normal subgroup of G_β , there is an induced homomorphism

$$\varphi: \mathrm{Aut} G_\beta \rightarrow \mathrm{Aut} \mathrm{PSL}(2, K),$$

which is easily seen to be an embedding (e.g., Robinson [4, 13.5.2]). Note that $\varphi \upharpoonright G_\beta$ is the identity map. So, identifying $\mathrm{Aut} G_\beta$ with its image under φ , it follows that $\mathrm{Aut} G_\beta = \mathrm{PGL}(2, K) \rtimes N_{\beta+1}(H(\alpha))$. Clearly no problems arise for limit ordinals. Thus, the automorphism tower of G terminates in exactly α steps. \square

Schenkman [5] proved that if L is a finite-dimensional centerless Lie algebra, then the tower of derivation algebras of L terminates after finitely many steps. A routine modification of the above argument shows that the tower of derivation algebras of an infinite-dimensional centerless Lie algebra L terminates in at most $(2^{\dim L})^+$ steps.

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