GROUPS ACTING ON THE RING OF TWO 2×2 GENERIC MATRICES AND A COPRODUCT DECOMPOSITION OF ITS TRACE RING

EDWARD FORMANEK¹ AND A. H. SCHOFIELD

ABSTRACT. Two results concerning the ring R generated by a pair of 2×2 generic matrices over a field K are proved: (1) The trace ring of R is a coproduct of commutative rings. (2) If a finite subgroup G of $\mathrm{SL}(2, K)$ acts homogeneously on R and the characteristic of K does not divide the order of G , then the fixed ring R^G is a finitely generated K -algebra.

1. Introduction. Let K be a field and let $X_1, \dots, X_r$ ($r \geq 2$) be $n \times n$ generic matrices over K . That is, $X_k = (x_{ij}(k))$ ($1 \leq i, j \leq n$), an $n \times n$ matrix whose entries are independent commuting variables over K . The K -subalgebra of $M_n([x_{ij}(k)])$ they generate is called a *ring of $n \times n$ generic matrices* and is denoted $R(n, r)$ or $K\{X_1, \dots, X_r\}$.

There are a number of subrings of $M_n(K[x_{ij}(k)])$ associated with $R(n, r)$. The subalgebra of $K[x_{ij}(k)]$ generated by the coefficients of the characteristic polynomials of all elements of $R(n, r)$ is called the *ring of invariants* of $R(n, r)$ and is denoted $\overline{C}(n, r)$. The subring of $M_n(K[x_{ij}(k)])$ generated by $R(n, r)$ and $\overline{C}(n, r)$ is called the *trace ring (or characteristic closure)* of $R(n, r)$ and is denoted $\overline{R}(n, r)$.

The ring $R(2, 2)$ generated by a pair of 2×2 generic matrices and the associated rings $\overline{C}(2, 2)$ and $\overline{R}(2, 2)$ enjoy several desirable properties not shared by other noncommutative generic matrix rings. For example, $\overline{C}(2, 2)$ is a polynomial ring over K and $\overline{R}(2, 2)$ is a free module over $\overline{C}(2, 2)$.

The object of this article is to prove two more results of a special nature about $R(2, 2)$. The first, Theorem 3, says that the trace ring $\overline{R}(2, 2)$ is a coproduct of polynomial rings over a polynomial ring. This almost certainly is not the case for any other noncommutative trace rings. Together with a result of Dicks [1], Theorem 3 implies that $\overline{R}(2, 2)$ has global dimension five (Corollary 4).

The second result concerns homogeneous actions of a finite group G on $R(n, r)$. These arise from linear actions of $G \leq \mathrm{GL}(r, K)$ on the K -vector space spanned by $X_1, \dots, X_r$. It is known that the fixed ring $R(n, r)^G$ is finitely generated if $R(n, r)$ is commutative [6, 7] (i.e. $n = 1$, so that $R(1, r)$ is a polynomial ring in r variables) or if G consists of scalar matrices in $\mathrm{GL}(r, K)$. The work of Fisher and Montgomery [4] suggests that $R(n, r)^G$ is not finitely generated for most (but not all) nonscalar actions if $n \geq 3$. Our Theorem 8 says that $R(2, 2)^G$ is finitely generated if $G \leq \mathrm{SL}(2, K)$ and the characteristic of K does not divide the order of G .

Received by the editors November 19, 1984.

1980 *Mathematics Subject Classification*. Primary 16A38.

¹Partially supported by the National Science Foundation and the John Simon Guggenheim Memorial Foundation.

©1985 American Mathematical Society
0002-9939/85 \$1.00 + \$.25 per page

A more detailed discussion of the ring of generic matrices and homogeneous group actions on it can be found in [2].

2. A coproduct decomposition of $\bar{R}(2, 2)$. From now on we will be concerned only with $R(2, 2)$, so we will usually write $R, \bar{C}, \bar{R}$ instead of $R(2, 2), \bar{C}(2, 2), \bar{R}(2, 2)$. In addition, the two generic matrices will be

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

LEMMA 1 [3, LEMMA 1]. (1) (*Multilinear Cayley-Hamilton Theorem*).

$$XY + YX - \text{tr}(Y)X - \text{tr}(X)Y + \text{tr}(X)\text{tr}(Y) - \text{tr}(XY) = 0.$$

(2) $\bar{C} = K[\text{tr}(X), \text{tr}(Y), \det(X), \det(Y), \text{tr}(XY)]$, a polynomial ring over K in 5 variables.

(3) $\bar{R} = \bar{C}I + \bar{C}X + \bar{C}Y + \bar{C}XY$, a free $\bar{C}$ -module of rank 4. $\square$

Define subrings A, P, Q of $\bar{R}$ by

$$A = K[\text{tr}(X), \text{tr}(Y), \det(X), \det(Y)],$$

$$P = K[\text{tr}(X), \text{tr}(Y), X, \det(Y)],$$

$$Q = K[\text{tr}(X), \text{tr}(Y), \det(X), Y].$$

LEMMA 2. (1) P and Q generate $\bar{R}$.

(2) A, P , and Q are commutative polynomial rings over K in 4 variables.

(3) $P = A \oplus AX$, $Q = A \oplus AY$, and both are free A -modules of rank 2.

(4) Let $Z = XY + YX - \text{tr}(X)Y - \text{tr}(Y)X$ in $P \amalg_A Q$. Then Z is central in $P \amalg_A Q$.

(5) $P \amalg_A Q = A[Z] + A[Z]X + A[Z]Y + A[Z]XY$.

PROOF. (1) Since $\bar{C}, X$ and Y generate $\bar{R}$ and all the generators of $\bar{C}$ except $\text{tr}(XY)$ lie in A , it suffices to show that $\text{tr}(XY)$ lies in the ring generated by A, X , and Y . This follows from the Multilinear Cayley-Hamilton Theorem. (2), (3), and (4) all follow from the usual Cayley-Hamilton Theorem, applied to X and Y . (5) Since A, X and Y generate $P \amalg_A Q$, the two sides of (5) will be equal provided that the right-hand side is a ring—i.e. it contains all products of pairs 1, X, Y, XY . This in turn follows from the Cayley-Hamilton Theorem and the equation

$$YX = Z + \text{tr}(Y)X + \text{tr}(X)Y - XY. \quad \square$$

THEOREM 3. The canonical map $\varphi: P \amalg_A Q \rightarrow \bar{R}(2, 2)$ is an isomorphism. Hence, $\bar{R}(2, 2)$ is the coproduct of two polynomial rings in four variables over K over a polynomial ring in four variables over K which has index 2 in both factors.

PROOF We will abuse notation slightly by not differentiating between $X, Y \in P \amalg_A Q$ and $X = \varphi(X), Y = \varphi(Y) \in \bar{R}$.

Since P and Q generate $\bar{R}$ by Lemma 2(1), φ is onto. Moreover, $\varphi(Z) = \text{tr}(XY) - \text{tr}(X)\text{tr}(Y)$ by the Multilinear Cayley-Hamilton Theorem, so

$$\varphi: A[Z] \rightarrow \bar{C} = A[\text{tr}(XY) - \text{tr}(X)\text{tr}(Y)]$$

is onto and hence an isomorphism, since $\text{tr}(XY) - \text{tr}(X)\text{tr}(Y)$ is transcendental over A . (Also, Z must be transcendental over A .)

Recall that, by Lemma 2(5) and Lemma 1(3),

$$\begin{aligned} P \amalg_A Q &= A[Z] + A[Z]X + A[Z]Y + A[Z]XY, \\ \overline{R} &= \overline{C}I + \overline{C}X + \overline{C}Y + \overline{C}XY, \end{aligned}$$

and $\overline{R}$ is a free $\overline{C}$ -module on I, X, Y, XY . Since $\varphi: A[Z] \rightarrow \overline{C}$ is an isomorphism, it follows that $1, X, Y, XY$ are free generators for $P \amalg_A Q$ as an $A[Z]$ -module and that $\varphi: P \amalg_A Q \rightarrow \overline{R}$ is an isomorphism. $\square$

Since A, P , and Q are polynomial rings in four variables over K , it follows from a result of Dicks [1, p. 567, Remark (ii) after Corollary 7] that $\overline{R} \cong P \amalg_A Q$ has global dimension less than or equal to five. On the other hand, since $\overline{R}$ is a free module over $\overline{C}$ and $\overline{C}$ is a polynomial ring in five variables over K (Lemma 1(2,3)), the global dimension of $\overline{R}$ is at least 5. I.e., let $K = \overline{R}/A$, where A is the ideal of elements of $\overline{R}$ of positive degree; then

$$\begin{aligned} \text{global dimension } (\overline{R}) &\geq \text{projective dimension of } K \text{ over } \overline{R} \\ &\geq \text{projective dimension of } K \text{ over } \overline{C} = 5. \end{aligned}$$

Thus we have the following result, which was proved by Small and Stafford [8] using iterated Ore extensions.

COROLLARY 4. *The global dimension of $\overline{R}(2, 2)$ is five. $\square$*

3. Homogeneous finite group actions on $R(2, 2)$. Let G be a finite subgroup of $\text{GL}(2, K)$. Then G acts homogeneously on $R = R(2, 2)$ and on R/J , where J is the commutator ideal of R . Note that R/J is a commutative polynomial ring over K in two variables.

LEMMA 5. (1) $(R/J)^G$ is a finitely generated K -algebra and R/J is a finitely generated $(R/J)^G$ -module.

(2) J/J^2 is a cyclic module over $(R/J) \otimes_K (R/J)$, where the action of $(R/J) \otimes_K (R/J)$ on J/J^2 is given by

$$((a + J) \otimes (b + J))(u + J^2) = auv + J^2.$$

(3) $(J/J^2)^G$ is a finitely generated module over $(R/J)^G \otimes_K (R/J)^G$.

PROOF. (1) is a basic result of E. Noether [6, 7].

(2) The commutator $XY - YX$ generates J as a two-sided ideal, so $(XY - YX) + J^2$ generates J/J^2 as a module over $(R/J) \otimes_K (R/J)$.

(3) By (1) and (2), J/J^2 is finitely generated—and thus Noetherian—as a module over $(R/J)^G \otimes_K (R/J)^G$. Hence the submodule $(J/J^2)^G$ is finitely generated as a module over $(R/J)^G \otimes_K (R/J)^G$. $\square$

LEMMA 6. *Suppose that $\text{char } K \nmid |G|$. Then $(R/J^2)^G$ is a finitely generated K -algebra.*

PROOF. Since $\text{char } K \nmid |G|$, the natural homomorphism $(R/J^2)^G \rightarrow (R/J)^G$ is onto, so there exist $a_1, \dots, a_l \in (R/J^2)^G$ (a finite set by Lemma 5(1)) whose images in $(R/J)^G$ generate $(R/J)^G$. By Lemma 5(3) there is a finite subset $b_1, \dots, b_m$ of $(J/J^2)^G$ which generates $(J/J^2)^G$ as a two-sided ideal of $(R/J^2)^G$. Then $a_1, \dots, a_l, b_1, \dots, b_m$ generate $(R/J^2)^G$ as a K -algebra. $\square$

REMARK. Lemma 6 is a special case of a result of Kharchenko [5, Theorem 1] because R/J^2 satisfies ACC on two-sided ideals.

LEMMA 7. $J^2 = J(XY - YX)$.

PROOF. By [3, Lemma 2] and right-left symmetry,

$$J = \overline{R}(XY - YX) = (XY - YX)\overline{R},$$

where $\overline{R}$ is the trace ring of R . Hence,

$$J^2 = \overline{R}(XY - YX)\overline{R}(XY - YX) = \overline{R}(XY - YX)(XY - YX) = J(XY - YX). \quad \square$$

THEOREM 8. *Let G be a finite subgroup of $\mathrm{SL}(2, K)$, where $\mathrm{char} K \nmid |G|$, and suppose that G acts homogeneously on $R = R(2, 2) = K\{X, Y\}$, the ring of two 2×2 generic matrices. Then R^G is a finitely generated K -algebra.*

PROOF. If $\mathrm{GL}(2, K)$ acts homogeneously on $R = K\{X, Y\}$ and $g \in \mathrm{GL}(2, K)$, then $(XY - YX)^g = \det(g)(XY - YX)$. Hence the hypothesis that $G \leq \mathrm{SL}(2, K)$ implies that $XY - YX \in R^G$.

There is an induced action of G on R/J^2 , and $(R/J^2)^G$ is a finitely generated K -algebra by Lemma 6. Since $\mathrm{char} K \nmid |G|$, the natural homomorphism $R^G \rightarrow (R/J^2)^G$ is onto, so there exist $a_1, \dots, a_m \in R^G$ whose images generate $(R/J^2)^G$ as a K -algebra.

Note that $R = K \oplus R_1 \oplus R_2 \oplus \dots$ is a graded domain. Moreover, the action of G preserves the grading, by hypothesis, so we may assume that $a_1, \dots, a_m$ are homogeneous.

We claim that $R^G = B$ where B is the K -algebra generated by $a_1, \dots, a_m$ and $XY - YX$. The claim is proved by induction on the degree of $u \in R^G$. Since the images of $a_1, \dots, a_m$ generate $(R/J^2)^G$ there is a polynomial $f(a_1, \dots, a_m) \in B$ such that $v = u - f(a_1, \dots, a_m)$ lies in J^2 . Since $a_1, \dots, a_m$ and J^2 are homogeneous, we can arrange that $\deg v \leq \deg u$. By Lemma 7, $v = w(XY - YX)$ for some $w \in J$. Since v and $XY - YX$ lies in R^G and $XY - YX$ is not a zero divisor (R is a domain). w also lies in R^G . Furthermore, either $v = w = 0$ or $\deg w < \deg v \leq \deg u$. Hence, $w \in B$ by the inductive hypothesis, so $v = w(XY - YX) \in B$, which proves the claim and the theorem. $\square$

REMARK. The group action $X \mapsto X, Y \mapsto -Y$ of $\mathbf{Z}_2$ on R shows that Theorem 8 cannot be extended to subgroups of $\mathrm{GL}(2, K)$.

The proof fails because J^2/J^3 is not finitely generated as a two-sided ideal (or R -bimodule). This is curious, because Lemma 2 of [3] implies that all J^n/J^{n+1} ($n \geq 1$) are isomorphic as left R -modules and as right R -modules. The explanation is that as R -bimodules

$$\begin{aligned} J/J^2 &\cong J^n/J^{n+1} & (n \text{ odd}), \\ J^2/J^3 &\cong J^n/J^{n+1} & (n \text{ even}), \end{aligned}$$

and the former is a free R -bimodule on one generator while the latter is infinitely generated as an R -bimodule.

REFERENCES

1. W. Dicks, *Mayer-Vietoris presentations over colimits of rings*, Proc. London Math. Soc. **34** (1977), 557–576.
2. E. Formanek, *Noncommutative invariant theory*, Group Actions on Rings (S. Montgomery, Editor), Contemporary Math. (to appear).
3. E. Formanek, P. Halpin and W.-C. W. Li, *The Poincaré series of the ring of 2×2 generic matrices*, J. Algebra **69** (1981), 105–112.

4. J. W. Fisher and S. Montgomery, *Invariants of finite cyclic groups acting on generic matrices*, J. Algebra (to appear).
5. V. K. Kharchenko, *Noncommutative invariants of finite groups and Noetherian varieties*, J. Pure Appl. Algebra **31** (1984), 83–90.
6. E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1916), 89–92. Reprinted in *Collected papers*, Springer-Verlag, New York, 1983, pp. 181–184.
7. ———, *Der Endlichkeitssatz der Invarianten endlicher Gruppen der Charakteristik p* , Nachr. v.d. Ges. d. Wiss. zu Göttingen (1926), 28–35. Reprinted in *Collected papers*, Springer-Verlag, New York, 1983, pp. 485–492.
8. L. W. Small and J. T. Stafford, *Homological properties of generic matrices*, Israel J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

DEPARTMENT OF MATHEMATICS, TRINITY COLLEGE, CAMBRIDGE, ENGLAND