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ON UNIMODULAR ROWS

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ABSTRACT. We prove here, among other results, that if $(x_0, \ldots, x_n)$ is a unimodular row over a commutative ring A, $n \geq 2$, $x \in A$ and

$$x \equiv x_n \mod J(Ax_0 + \cdots + Ax_{n-2}),$$

then $(x_0, \ldots, x_{n-1}, x_n) \sim_E (x_0, \ldots, x_{n-1}, x)$.

This note is based on Suslin's work on projective modules (see [**1** and **3**]). Among other results, we simplify the proof of Suslin's theorem concerning the completability of the unimodular row $(x_0^{r_0}, \ldots, x_n^{r_n})$, when $n! | \prod_{i=0}^n r_i$. More precisely, we simplify the reduction to the case $(x_0, x_1, x_2^2, \ldots, x_n^n)$.

All the rings here are commutative with unit. We denote by $U_n(A)$ the set of unimodular rows of length n over the ring A. If u, v are elements in $U_n(A)$, we denote $u \sim v$ if there exists a matrix α in $\mathrm{GL}_n(A)$ such that $u\alpha = v$. We have $u \sim (1, 0, \ldots, 0)$ if and only if u is completable to a matrix in $\mathrm{GL}_n(A)$. Similarly, we denote $u \sim_E v$ if there exists a matrix α in $E_n(A)$ such that $u\alpha = v$.

If I is an ideal in a ring A, we denote by $J(I)$ the intersection of all the maximal ideals in A containing I.

LEMMA 1. *Let A be a ring, $(x_0, \ldots, x_n) \in U_{n+1}(A)$, $n \geq 2$, and let y_{n-1}, y_n be elements of A such that $x_{n-1}y_{n-1} + x_n y_n$ is invertible modulo the ideal $Ax_0 + \cdots + Ax_{n-2}$. Then $(x_0, \ldots, x_{n-2}, x_{n-1}, x_n) \sim_E (x_0, \ldots, x_{n-2}, y_{n-1}, y_n)$.*

PROOF. Let $z_1, \ldots, z_{n-2}, t$ be elements of A such that

$$\left(\sum_{i=0}^{n-2} x_i z_i\right) + t(x_{n-1}y_{n-1} + x_n y_n) = 1.$$

Let $z_{n-1} = t(y_{n-1} + x_n)$, $z_n = t(y_n - x_{n-1})$ and $u = (z_0, \ldots, z_n)$. We have

$$(x_0, \ldots, x_{n-2}, x_{n-1}, x_n)u^t = (x_0, \ldots, x_{n-2}, y_n, -y_{n-1})u^t = 1.$$

By [**3**, Corollary 2.8] we obtain

$$\begin{aligned}(x_0, \ldots, x_{n-2}, x_{n-1}, x_n) &\sim_E (x_0, \ldots, x_{n-2}, y_n, -y_{n-1}) \\ &\sim_E (x_0, \ldots, x_{n-2}, y_{n-1}, y_n). \quad \square\end{aligned}$$

THEOREM 2. *Let $(x_0, \ldots, x_n) \in U_{n+1}(A)$, $n \geq 2$, let*

$$I_j = J(Ax_0 + \cdots + Ax_{j-1} + Ax_{j+1} + \cdots + Ax_{n-1})$$

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for $0 \leq j \leq n-1$, *and let* $x \in A$, *such that* $x \equiv x_n \bmod (I_0 + \cdots + I_{n-1})$. *Then* $(x_0, \ldots, x_{n-1}, x_n) \sim_E (x_0, \ldots, x_{n-1}, x)$.

PROOF. Let $x = x_n + t_0 + \cdots + t_{n-1}$, where $t_j \in I_j$ $(0 \leq j \leq n-1)$. It is enough to prove, for $0 \leq j \leq n-1$, that

$$(x_0, \ldots, x_{n-1}, x_n + t_0 + \cdots + t_{j-1}) \sim_E (x_0, \ldots, x_{n-1}, x_n + t_0 + \cdots + t_j),$$

so we may assume $x \equiv x_n \bmod J(Ax_0 + \cdots + Ax_{n-2})$. Let y_{n-1}, y_n be such that $x_{n-1}y_{n-1} + x_n y_n \equiv 1 \bmod (Ax_0 + \cdots + Ax_{n-2})$. Then

$$x_{n-1}y_{n-1} + xy_n \equiv x_{n-1}y_{n-1} + x_n y_n \mod J(Ax_0 + \cdots + Ax_{n-2}),$$

so $x_{n-1}y_{n-1} + xy_n$ is invertible $\bmod (Ax_0 + \cdots + Ax_{n-2})$. By Lemma 1 we have

$$(x_0, \ldots, x_{n-1}, x_n) \sim_E (x_0, \ldots, x_{n-2}, y_{n-1}, y_n) \sim_E (x_0, \ldots, x_{n-2}, x_{n-1}, x). \quad \square$$

We do not know if the assumption of Theorem 2 may be replaced by the assumption $x \equiv x_n \bmod J(Ax_0 + \cdots + Ax_{n-1})$ or even by the assumption $x \equiv x_n \bmod \sqrt{Ax_0 + \cdots + Ax_{n-1}}$ (see Propositions 5 and 6 below).

As pointed out by L. N. Vaserstein, Proposition 3 (and also the simplification in the proof of Suslin's theorem below) was already obtained in other ways in [**5**] and also by R. A. Rao-M. P. Murthy.

PROPOSITION 3 (CF. [**3**, §2 OR **1**, CHAPTER V]). *Let* $(x_0, \ldots, x_n) \in U_{n+1}(A)$, $n \geq 2$, *and let* $r_0, \ldots, r_n, r'_0, \ldots, r'_n$ *be natural numbers such that* $\prod_{i=0}^n r_i = \prod_{i=0}^n r'_i = r$. *Then* $(x_0^{r_0}, \ldots, x_n^{r_n}) \sim_E (x_0^{r'_0}, \ldots, x_n^{r'_n}) \sim_E (x_0^r, x_1, \ldots, x_n)$.

PROOF. By Theorem 2, we have, for any $s \geq 0$, that

$$(x_0^s, x_1, \ldots, x_n) \sim_E (x_0^s, x_1 - x_0, \ldots, x_n), \quad \text{as } x_1 \equiv x_1 - x_0 \mod \sqrt{Ax_0^s}.$$

Furthermore, $(x_0^s, x_1 - x_0, \ldots, x_n) \sim_E (x_1^s, x_1 - x_0, \ldots, x_n) \sim_E (x_1^s, -x_0, \ldots, x_n) \sim_E (x_0, x_1^s, \ldots, x_n)$ so the proposition follows. $\square$

THEOREM 4 (SUSLIN [**5**, THEOREM 2]). *Let* $(x_0, \ldots, x_n) \in U_{n+1}(A)$, $n \geq 2$, *and let* $r_0, \ldots, r_n$ *be natural numbers such that* $n! \mid \prod_{i=0}^n r_i$. *Then,* $(x_0^{r_0}, \ldots, x_n^{r_n}) \sim (1, 0, \ldots, 0)$.

PROOF. Let $\prod_{i=0}^n r_i = n!d$. Then, by Proposition 3 and [**3**, Proposition 1.6], we have $(x_0^{r_0}, \ldots, x_n^{r_n}) \sim_E (x_0^d, x_1^d, (x_2^d)^2, \ldots, (x_n^d)^n) \sim (1, 0, \ldots, 0)$. $\square$

If u, v are in $U_n(A)$ we denote $u \leftrightarrow v$ for the property $u \sim (1, 0, \ldots, 0)$ if and only if $v \sim (1, 0, \ldots, 0)$.

PROPOSITION 5. *For any ring* A *and* $n \geq 2$, *the following conditions are equivalent:*

(1) *For any* $(x_0, \ldots, x_n)$ *in* $U_{n+1}(A)$ *and* $x \in A$ *such that*

$$x \equiv x_n \mod \sqrt{Ax_0 + \cdots + Ax_{n-1}},$$

we have $(x_0, \ldots, x_{n-1}, x_n) \leftrightarrow (x_0, \ldots, x_{n-1}, x)$.

(2) *If* $x_0, \ldots x_n$ *are elements of* A *such that* x_n *is unipotent*

$$\mod (Ax_0 + \cdots + Ax_{n-1}),$$

then $(x_0, \ldots, x_n) \sim (1, 0, \ldots, 0)$.

PROOF. (1)⇒(2). We have

$$x_n \equiv 1 \bmod \sqrt{Ax_0 + \cdots + Ax_{n-1}},$$

so $(x_0, \ldots, x_{n-1}, x_n) \leftrightarrow (x_0, \ldots, x_{n-1}, 1)$. As $(x_0, \ldots, x_{n-1}, 1) \sim_E (1, 0, \ldots, 0)$ we obtain $(x_0, \ldots, x_n) \sim (1, 0, \ldots, 0)$.

(2)⇒(1). Assume $(x_0, \ldots, x) \sim (1, 0, \ldots, 0)$. Let $y \in A$ such that $xy \equiv 1$ mod $(Ax_0 + \cdots + Ax_{n-1})$. Then $(x_0, \ldots, x_{n-1}, x_n) \sim_E (x_0, \ldots, x_{n-1}, x_n(xy)) = (x_0, \ldots, x_{n-1}, x(x_ny))$. But $x_ny \equiv xy \equiv 1 \bmod \sqrt{Ax_0 + \cdots + Ax_{n-1}}$, so x_ny is unipotent mod$(Ax_0 + \cdots + Ax_{n-1})$. By assumption, for any generators $z_0, \ldots, z_{n-1}$ of the ideal $Ax_0 + \cdots + Ax_{n-1}$, we have $(z_0, \ldots, z_{n-1}, x_ny) \sim (1, 0, \ldots, 0)$, so $(x_0, \ldots, x_{n-1}, x(x_ny)) \sim (1, 0, \ldots, 0)$ by [**3**, Corollary 3.3]. Finally,

$$(x_0, \ldots, x_{n-1}, x_n) \sim (1, 0, \ldots, 0). \qquad \square$$

PROPOSITION 6. *The two properties of Proposition* 5 *hold in each of the following cases*:

(1) *$n!$ is invertible in A.*

(2) *Stably free $A[X]$-modules are extended from A.*

(1) (See [**3**, Proposition 3.1 or **4**, Theorem 1.6] and their proofs .) We prove property 2 of Proposition 5. Let $(x_0, \ldots, x_n) \in U_{n+1}(A)$,

$$x_n \equiv 1 \quad \bmod \sqrt{Ax_0 + \cdots + Ax_{n-1}},$$

$(x_n - 1)^k \in Ax_0 + \cdots + Ax_{n-1}$, $k \geq 1$. As $n!$ is invertible in A, by a standard argument we have $x_n \equiv y^{n!} \bmod (Ax_0 + \cdots + Ax_{n-1})$, where

$$y = \sum_{i=0}^{k-1} \binom{1/n!}{i} (x_n - 1)^i.$$

It follows that $(x_0, \ldots, x_{n-1}) \sim_E (x_0, \ldots, x_{n-1}, y^{n!}) \sim (1, 0, \ldots, 0)$. $\square$

(2) Let $(x_0, \ldots, x_n) \sim U_{n+1}(A)$, $x_n = 1+a$, $a \in \sqrt{Ax_0 + \cdots + Ax_{n-1}}$. Then the row $u(X) = (x_0, \ldots, x_{n-1}, 1+aX)$ is unimodular over $A[X]$. It follows from our assumption that $u(X) \sim u(0) = (1, 0, \ldots, 0)$. Therefore, we have, over A, $(x_0, \ldots, x_n) = u(1) \sim (1, 0, \ldots, 0)$. $\square$

Proposition 6(2) may be applied to regular affine algebras (by Lindel's theorem, etc.). By the argument in the proof of Proposition 6(2) we see that an example of a noncompletable unimodular row $(x_0, \ldots, x_n)$ over a ring A, with x_n unipotent mod $(Ax_0 + \cdots + Ax_{n-1})$, would also provide an example of a stably free $A[X]$-module which is not extended from A (see [**4**, p. 114 and **2**, Chapter V.3, p. 140].

The next proposition is a direct consequence of Lemma 1.

PROPOSITION 7. *If u, v are in $U_n(A)$, n even and uv^t is an invertible element of A, them $u \sim_E v$.*

As shown by L. N. Vaserstein, Proposition 7 does not hold for odd $n \geq 3$ (take $A = \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2 - 1)$; $u = (\overline{x}_1, \ldots, \overline{x}_n)$, $v = -u$). Anyway for any $n \geq 1$ and u, v in $U_n(A)$ such that $uv^t = c$ is invertible in A, we have $u \leftrightarrow v$. (If u is the first row of $\alpha \in \mathrm{GL}_n(A)$, then $v\alpha^t = (c, \ldots) \sim (1, 0, \ldots, 0)$, so $v \sim (1, 0, \ldots, 0)$.) We generalize Lemma 1 and its proof.

THEOREM 8 (CF. [**3**, LEMMA 2.2 OR **1**, CHAPTER V, LEMMA 2.3]). *Let $(x_0, \dots, x_n) \in U_n(A)$, $n \geq 2$ and let $0 \leq k \leq n-1$, $y_i \in A$ $(k \leq i \leq n)$. Let I be the ideal in A generated by the 2×2 minors of the matrix*

$$\alpha = \begin{pmatrix} x_k \cdots x_n \\ y_k \cdots y_n \end{pmatrix}.$$

Assume $Ax_0 + \cdots + Ax_{k-1} + I = A$. Then

$$(x_0, \dots, x_{k-1}, x_k, \dots, x_n) \sim_E (x_0, \dots, x_{k-1}, y_k, \dots, y_n).$$

PROOF. Let $d \in I$, $a_i \in A$ $(0 \leq i \leq k-1)$ such that $1 = d + \sum_{i=0}^{k-1} a_i x_i$. There exists an $(n-k+1) \times 2$ matrix β such that $\alpha\beta = dI_2$ (see [**1**, Chapter V, Lemma 2.2]; our formulation is obtained by transposing). Let

$$\alpha' = \begin{pmatrix} a_k \\ \vdots \\ a_n \end{pmatrix}$$

be the sum of the two columns of β. Then $\alpha\alpha' = \binom{d}{d}$, so $\sum_{i=k}^{n} x_i a_i = \sum_{i=k}^{n} y_i a_i = d$. Let $u = (a_0, \dots, a_k, a_{k+1}, \dots, a_n)$. Then

$$(x_0, \dots, x_{k-1}, x_k, \dots, x_n)u^t = (x_0, \dots, x_{k-1}, y_k, \dots, y_n)u^t = 1,$$

so $(x_0, \dots, x_n) \sim_E (x_0, \dots, x_{k-1}, y_k, \dots, y_n)$ by [**3**, Corollary 2.8]. $\square$

In the formulation of Theorem 8, let $\overline{A} = A/(Ax_0 + \cdots + Ax_{k-1})$. The condition $Ax_0 + \cdots + Ax_{k-1} + I = A$ (the unimodularity of the matrix $\overline{\alpha}$ over $\overline{A}$) is equivalent to each of the following two conditions:

(i) The matrix $\overline{\alpha}$ over $\overline{A}$ has a right inverse.

(ii) $((\overline{x}_k, \dots, \overline{x}_n), (\overline{y}_k, \dots, \overline{y}_n))$ is a free basis of a direct summand of the $\overline{A}$-module $\overline{A}^{n-k+1}$. For $k = -1$, Theorem 8 generalizes [**3**, Corollary 2.9].

REMARK. The main result of [**5**] (which generalizes Proposition 3 above) may be easily obtained using the results above.

THEOREM (L. N. VASERSTEIN). *Let $u = (x_0, \dots, x_n), v = (y_0, \dots, y_n)$ be in $U_{n+1}(R), n \geq 2, u \sim_E v$. Then for any $m \geq 1$ we have $(x_0^m, x_1, \dots, x_n) \sim_E (y_0^m, y_1, \dots, y_n)$. If*

$$x_0'x_0 \equiv 1 \pmod{Rx_1 + \cdots + Rx_n}, \qquad y_0'y_0 \equiv 1 \pmod{Ry_1 + \cdots + Ry_n},$$

then $(x_0', x_1, \dots, x_n) \sim_E (y_0', y_1, \dots, y_n)$.

PROOF. As Proposition 3 above allows moving exponents from one entry to another, to prove $(x_0^m, \dots, x_n) \sim_E (y_0^m, \dots, y_n)$, it is enough to show $(x_0^m, \dots, x_n) \sim_E (y_0^m, \dots, y_n)$, where $y_1 = x_1 + rx_2$ for some $r \in R, y_i = x_i$ for $0 \leq i \leq n, i \neq 1$ and this is obvious.

Similarly, to obtain $(x_0', \dots, x_n) \sim_E (y_0', \dots, y_n)$, it is enough to show that if $x_i'x_i \equiv 1 \pmod{\sum_{k \neq i} Rx_k}$, $x_j'x_j \equiv 1 \pmod{\sum_{k \neq j} Rx_k}$, then $(x_0, \dots, x_i', \dots, x_n) \sim_E (x_0, \dots, x_j', \dots, x_n)$. We may assume $i = 0, j = 1$ and by correcting x_1' if necessary we may assume also $\sum_{k=0}^{n} x_k x_k' = 1$, for some x_k' $(2 \leq k \leq n)$ in R.

By Lemma 1 above we have $(x_0', x_1, \dots, x_n) \sim_E (x_0, x_1', \dots, x_n)$ because $x_0'x_0 + x_1x_1' \equiv 1 \pmod{Rx_2 + \cdots + Rx_n}$. $\square$

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