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THE PROOF OF A CONJECTURE OF GRAHAM FOR SEQUENCES CONTAINING PRIMES[1]

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ABSTRACT. Let $a_1 < a_2 < \cdots < a_n$ be a finite sequence of positive integers. R. L. Graham has conjectured that $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$. We verify this conjecture in case at least one of the a_i's is prime.

R. L. Graham [**2**] has conjectured that if $a_1 < a_2 < \cdots < a_n$ is a sequence of positive integers, then $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$. The conjecture has been verified in some special cases. For references see [**1**]. We mention here (i) the case when a_1 is a prime [**5**] and (ii) the case when, for some k, a_k is a prime not being of the form $p = \frac{1}{2}(a_i + a_j)$ [**4**].

In this note we prove Graham's conjecture for sequences containing a prime. Thus we obtain the above result (ii) without any restriction on p.

THEOREM. *Let $a_1 < a_2 < \cdots < a_n$ be a sequence of positive integers where $a_k = p$, a prime, for some k. Then* $\max_{i,j}\{a_i/(a_i, a_j)\} \geq n$.

PROOF. Assume the contrary that $\max_{i,j}\{a_i/(a_i, a_j)\} < n$. Since we may suppose that g.c.d.$\{a_1, \ldots, a_n\} = 1$, some a_i is not a multiple of p, so $p = a_k = a_k/(a_k, a_i) < n$. Our sequence contains elements $\geq n$ so $a_1 > 1$. Moreover, each $a_j \geq n$ must be a multiple of p since otherwise $a_j/(a_j, a_k) = a_j \geq n$. In particular, if $a_n = tp$ then $t = a_n/(a_n, a_k) \leq n - 1$. We claim that $t \leq n - 2$. Indeed, by [**4**, **5**], a_1 is not prime, so $k > 1$ and $a_1 < a_k = p$. It follows that $a_1 \geq 4$ and $p \geq 5$. Since $a_n/(a_n, a_1) \leq n - 1$, we have $a_n \leq (n-1)a_1$, so $tp \leq (n-1)a_1 < (n-1)p$ and this implies that $t \leq n - 2$.

Consider the following two sets:

$$A = \{1, 2, \ldots, n-1\}, \quad B = \{sp, (s+1)p, \ldots, tp\}$$

where $s = \lceil n/p \rceil$. By the definitions of s, t and B each $a_j \geq n$ belongs to B, so $\{a_1, \ldots, a_n\} \subseteq A \cup B$.

The proof will be achieved by defining a 1-1 correspondence $F\colon B \to A$ such that if $F(b) = a$ then at most one of a, b can be a member of $C = \{a_1, \ldots, a_n\}$. Hence C has at most $n - 1$ elements, a contradiction.

Let $A' = \{c \in A | p \nmid c\}$. If $x \in B$ then $x = bp^r$ for some $b, p \nmid b$, and $b \leq n-2$, $r \geq 1$. Define $f\colon B \to A'$ by $f(x) = b$. The function f is 1-1 for if $f(x') = f(x'')$ and $x' \neq x''$, say $x'' > x'$, let $x' = b'p^{r'}$, $p \nmid b'$, $x'' = b''p^{r''}$, $p \nmid b''$. Then $b' = f(x') = f(x'') = b''$ and $r'' > r'$ so $x'' = p^{r'' - r'}x' \geq pn > pt$, which is impossible.

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Now let $B' = f(B)$. Then $B' \subset A'$. If $b \in B'$ then $p \nmid b$ so $b \equiv u \pmod p$ for some u, $1 \le u \le p-1$. Define $g \colon B' \to A'$ as follows

$$g(b) = \begin{cases} b+1 & \text{if } u \text{ is odd,} \\ b-1 & \text{if } u \text{ is even.} \end{cases}$$

The values of g belong to A' since $p-1$ is even and $t \le n-2$. The function g is clearly also 1-1.

The required correspondence F from B to A is defined by $F(x) = g(f(x))$ and it is clearly 1-1. We claim that $F(x)$ and x are relatively prime. Indeed, $F(x) \in A'$ so $p \nmid F(x)$. In addition, $x = bp^r$ and $F(x) = b \pm 1$, so $(b, F(x)) = 1$, it follows that $(bp^r, F(x)) = 1$, thus $(x, F(x)) = 1$. This implies that x and $F(x)$ cannot both belong to $\{a_1, \dots, a_n\}$ because $x/(x, F(x)) = x \ge n$, and this proves the theorem.

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