PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 2, October 1985

RATIONAL ALGEBRAIC K-THEORY OF CERTAIN TRUNCATED POLYNOMIAL RINGS

R. E. STAFFELDT[1]

ABSTRACT. In this paper we derive a formula for rationalized algebraic K-theory of certain overrings of rings of integers in number fields. Truncated polynomial algebras are examples. Our method is homological calculation which is facilitated by some basic rational homotopy theory and interpreted in terms of the cyclic homology theory of algebras invented by Alain Connes.

The object of this paper is to compute, in terms of A. Connes' cyclic homology and the rational algebraic K-theory of a ring of integers $\mathcal{O}$ in a number field $k = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Q}$, the rationalized K-theory of an augmented $\mathcal{O}$-algebra $A \xrightarrow{\varepsilon} \mathcal{O}$ which is finitely generated projective as an $\mathcal{O}$-module and whose augmentation ideal $\bar{A}$ is nilpotent. The standard example to have in mind is $A = \mathcal{O}[T]/(T^{n+1})$, a truncated polynomial algebra. Our main result may be stated as follows:

THEOREM 1. *For $q \geqslant 1$ and A as above,*

$$K_q(A) \otimes \mathbf{Q} \cong K_q(\mathcal{O}) \otimes \mathbf{Q} \oplus V_q$$

where V_q is a rational vector space of dimension $d \cdot \dim_k HC_{q-1}(k \otimes_{\mathcal{O}} \bar{A})$ and d is the degree of k over $\mathbf{Q}$.

This formula for $K_q(A) \otimes \mathbf{Q}$ follows that obtained by C. Soulé in [**9**] for the case $A = \mathcal{O}[T]/(T^2)$, the dual numbers over $\mathcal{O}$. Using a classical Lie algebra homology calculation based on invariant theory, but no cyclic homology, he found that for the dual numbers $\dim_{\mathbf{Q}} V_q = d$, or $= 0$, if q is odd, or even. In §2 of this paper we sketch a computation in cyclic homology relevant to the case $A = \mathcal{O}[T]/(T^{n+1})$ following the steps of a computation of the rational algebraic K-theory of the space $\mathbf{CP}^n$ shown to us by T. Goodwillie. It turns out that $\dim V_q = n \cdot d$, or $= 0$, depending on whether or not q is odd, or even.

The proof of this theorem, together with the proof of the theorem of Loday and Quillen relating Lie algebra homology and cyclic homology, actually gives when $\mathbf{Z} = \mathcal{O}$ a chain of natural isomorphisms linking the rationalized relative K-group

$$K_{q+1}(A \xrightarrow{\varepsilon} \mathbf{Z}) = \pi_q(\text{fibre}(\mathrm{BGL}^+(A) \to \mathrm{BGL}(\mathbf{Z})^+)) \otimes \mathbf{Q}$$

Received by the editors May 18, 1984 and, in revised form, December 28, 1984.
1980 *Mathematics Subject Classification*. Primary 18F25, 55P62.
[1] Partially supported by NSF grant MSC-80-002 396.

0002-9939/85 $1.00 + $.25 per page

with the rationalized cyclic homology $H\bar{C}_{q-1}(\mathbf{Q} \otimes \bar{A}) \cong HC_q(\mathbf{Q} \otimes A \to \mathbf{Q})$. Philosophically, there is nothing really special about the ring $\mathbf{Z}$ in the K-theoretic context. Our proof using rational homotopy theory puts the field $\mathbf{Q}$ in a privileged role. This seems to imply that the definition of cyclic homology and the Loday–Quillen proof will have to be refined somewhat to take into account rationality questions. We prefer not to try to introduce such technicalities into this account and our present result, therefore, has its slightly unnatural flavor.

As a second remark, the groups $\mathrm{GL}_n(A)$ are arithmetic, so that their integral homology groups are finitely generated in each degree. A is a ring to which van der Kallen's stability results **[10]** apply, so that in the limit the integral homology of $\mathrm{GL}(A)$ is finitely generated. Hence, the K-groups of A are also finitely generated, by Serre's generalization of the Hurewicz theorem.

The main part of the proof is in §1, and is an application of rational homotopy theory. §2 contains a definition of cyclic homology and quotes results that lead to the numerical examples.

I would also like to thank W. G. Dwyer for the discussion that led to this paper.

1. We here adopt the plus-construction definition of algebraic K-theory and study the spectral sequences of the fibrations of classifying spaces associated to the exact sequences of groups

$$1 \to \mathrm{G}_n(\bar{A}) \to \mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathcal{O}) \to 1.$$

It will be seen that the group $\mathrm{G}_n(\bar{A})$ is nilpotent, so rational homotopy theory works to supply a dual Lie algebra model for the classifying space $\mathrm{BG}_n(\bar{A})$. It is then possible to analyze the spectral sequences and to complete the proof. In general outline, the approach is very similar to the approach to the rational algebraic K-theory of topological spaces in **[2]**.

We are taking as our definition $K_i(A) = \pi_i \mathrm{BGL}^+(A)$ for $i \geqslant 1$. By the Milnor-Moore theorem,

$$\pi_i(\mathrm{BGL}^+(A)) \otimes \mathbf{Q} \cong \mathrm{Prim}\, H_i(\mathrm{BGL}^+(A); \mathbf{Q}) \cong \mathrm{Prim}\, H_i(\mathrm{BGL}(A); \mathbf{Q}),$$

since $\mathrm{BGL}(A) \to \mathrm{BGL}^+(A)$ induces a homology isomorphism. We will extract Theorem 1 from the spectral sequence of the fibration

$$\mathrm{BG}(\bar{A}) \to \mathrm{BGL}(A) \to \mathrm{BGL}(\mathcal{O})$$

which is obtained from the exact sequences of groups

$$1 \to \mathrm{G}_n(\bar{A}) \to \mathrm{GL}_n(A) \xrightarrow{\varepsilon} \mathrm{GL}_n(\mathcal{O}) \to 1$$

by passage to limit followed by passage to classifying spaces. ε is the map induced by augmentation $A \to \mathcal{O}$, so $\mathrm{G}_n(\bar{A})$ is therefore the group of matrices over A of the form $I_n + M$ where I_n is the $n \times n$ identity matrix and $M \in M_n(\bar{A}) \subset M_n(A)$. Recall that the lower central series filtration on G is defined by $\Gamma_0 G = G$, and, for $i \geqslant 1$, $\Gamma_i G = [G, \Gamma_{i-1} G]$. A brief calculation gives $\Gamma_i \mathrm{G}_n(\bar{A}) \subset I_n + M_n(\bar{A}^{i+1})$ so our hypothesis that $\bar{A}^N = 0$ for N sufficiently large implies that $\mathrm{G}_n(\bar{A})$ is nilpotent. Note also that it is clear from the definition that $G_n(\bar{A})$ is torsion-free and that one may prove that $G_n(\bar{A})$ is finitely-generated by considering the sequences

$$1 \to \left(I_n + M_n\left(\bar{A}^m/\bar{A}^{m+1}\right)\right) \to G_n\left(\bar{A}/\left(\bar{A}^{m+1}\right)\right) \to G_n\left(\bar{A}/\left(\bar{A}\right)^m\right) \to 1$$

for $m = 1, 2, \ldots, N-1$.

Therefore rational homotopy theory will work quite well when applied to the classifying space $BG_n(\bar{A})$. We will need to interpret the following general facts which are taken from Chapter XII of **[4]**. First, if X is a $K(\pi, 1)$ complex with $\pi_1 = \pi$ finitely generated and nilpotent, then any **Q**-algebra of commutative cochains on X has a 1-minimal model $\mathscr{M}_1 \xrightarrow{\rho_1} A(X)$ which is unique up to homotopy. ($\mathscr{M}_1$ is a commutative differential graded algebra with decomposable differential and ρ_1 induces a (co)homology isomorphism.) Second, if $f\colon X \to Y$ is a basepoint preserving map, it induces a well-defined map of minimal models $\hat{f}\colon \mathscr{M}_{Y,1} \to \mathscr{M}_{X,1}$. Finally, when X is as above, $(\mathscr{M}_1, d)$ is dual to the tower of **Q**-Lie algebras associated to $\pi \otimes \mathbf{Q}$, which is π made uniquely divisible. In particular, $(\mathscr{M}_1, d)$ turns out to be the Koszul complex for computing Lie algebra cohomology with **Q** coefficients for the Lie algebra associated by Mal′cev to the torsion-free, uniquely divisible group $\pi \otimes \mathbf{Q}$. The greatest part of our work is to make this last remark explicit in the case at hand.

THEOREM 2. *The associated Lie algebra of* $G_n(\bar{A})$ *is* $M_n(\bar{A} \otimes \mathbf{Q})$, *the Lie algebra of* $n \times n$ *matrices over the ring without unit* $\bar{A} \otimes \mathbf{Q} = \bar{R}$. *Moreover, the naturality property of the* 1-*minimal model implies the additional fact that* $\Lambda^* M_n(\bar{A} \otimes \mathbf{Q}) \to A(BG_n(\bar{A}))$ *is* $GL_n(\mathscr{O})$ *equivariant, where* $GL_n(\mathscr{O})$ *acts on* $M_n(\bar{A} \otimes \mathbf{Q})$ *by conjugation.*

Assuming this result we can derive Theorem 1 from the three parts of

THEOREM 3. *Consider the fibration*

$$\text{(I)} \qquad BG(\bar{A}) \to BGL(A) \to BGL(\mathscr{O}),$$

let $R = \mathbf{Q} \otimes_{\mathbf{Z}} A = k \otimes_{\mathscr{O}} A$, *and let* $\bar{R}$ *be the augmentation ideal of* R. *Recall that* $k = \mathbf{Q} \otimes_{\mathbf{Z}} \mathscr{O}$.

(i) *For the rational homology spectral sequence of* (I),

$$E^2_{**} \cong H_*(GL(\mathscr{O}); \mathbf{Q}) \otimes H_*\left(M(\bar{R}); \mathbf{Q}\right)_{GL(k)}.$$

(ii) *This spectral sequence collapses at* E^2.

(iii) Prim $H_*(BGL(A); \mathbf{Q}) \cong$ Prim $H_*(BGL(\mathscr{O}); \mathbf{Q}) \oplus (V_*)^d$, *where d is the degree of k over* **Q** *and* $\dim_{\mathbf{Q}} V_* = \dim_k HC_{*-1}(\bar{R})$.

PROOF OF (i). Dualizing Theorem 2, a $GL_n(\mathscr{O})$-equivariant model for the rational chains on $BG_n(\bar{A})$ is furnished by the rational Koszul complex $\Lambda_*(M_n(\bar{R}))$ and the action of the fundamental group of the base on the fibre has gone over to the action on Λ_* induced by conjugation of matrices. So

$$H_*\left(BGL_n(\mathscr{O}); H_*\left(BG_n(\bar{A}); \mathbf{Q}\right)\right) \cong H_*\left(GL_n(\mathscr{O}); H_*\left(M_n(\bar{R}); \mathbf{Q}\right)\right).$$

To be more precise we should make use of the reduction of scalars construction $R_{k/\mathbf{Q}}$ **[11]**. Applied to $GL_n(k)$ this furnishes us with an algebraic group G_n whose **Q** points are $GL_n(k)$. Finally, the rational $GL_n(k)$-Lie algebra $M_n(\bar{R})$ is really the $G_n(\mathbf{Q})$-Lie algebra $R_{k/\mathbf{Q}}(M_n(\bar{R}))$.

Now each complex $\Lambda_*(M_n(\overline{R}))$ is a complex of algebraic representations of $\mathrm{G}_n(\mathbf{Q}) \cong \mathrm{GL}_n(k)$ which splits into subcomplexes along representation types according to Schur's lemma. Notably, the quotient complex of covariants $\Lambda_*(M_n(\overline{R}))_{\mathrm{GL}_n(k)}$ captures the trivial representations, and in the limit nothing else contributes to the spectral sequence of the fibration (I). For the representations involved in $\Lambda_* M_n(\overline{R})$ are all subrepresentations of $\otimes^M M_n(\overline{R})$, a tensor power over $\mathbf{Q}$. Complexifying this space and the group G_n gives $\otimes^M(\oplus^d M_n(\mathbf{C}))$ acted on by factorwise conjugation by $(\mathrm{GL}_n(\mathbf{C}))^d$. The invariant multilinear functions corresponding to trivial representations of $\mathrm{GL}_n(\mathbf{C})$ in $\otimes M_n(\mathbf{C})$ are just products of traces of products of matrices. When n is large compared to the number of factors there are no other possibilities for $\mathrm{SL}_n(\mathbf{C})$ invariants [**7**, **12**]. Applying this fact inductively and retreating to $\mathrm{SL}_n(k) \subset \mathrm{GL}_n(k)$, we see that stably the $\mathrm{SL}_n(k)$- and $\mathrm{GL}_n(k)$-trivial subspaces are the same. Then the following vanishing theorem applies: If SG_n is the algebraic group whose $\mathbf{Q}$-points are $\mathrm{SL}_n(k)$ and whose $\mathbf{Z}$-points are $\mathrm{SL}_n(\mathcal{O})$ and ρ is a nontrivial irreducible algebraic representation of $\mathrm{SG}_n(\overline{\mathbf{R}})$, then $H_p(\mathrm{SL}_n(\mathcal{O});\, \rho) = 0$ for p in a stable range $0 \leqslant p < \varphi(n)$ where $\varphi(n)$ tends to infinity with n [**1**, **3**]. Then

$$\begin{aligned}
&H_*\big(\mathrm{BGL}(\mathcal{O});\, H_*(\mathrm{BG}(\overline{A}))\big)\\
&\quad = \operatorname{ind\,lim} H_*\big(\mathrm{BGL}_n(\mathcal{O});\, H_*(\mathrm{BG}_n(\overline{A}))\big)\\
&\quad \cong \operatorname{ind\,lim} H_*\big(\mathrm{GL}_n(\mathcal{O});\, H_*(M_n(\overline{R}))\big)\\
&\quad \cong \operatorname{ind\,lim} H_*\big(\mathrm{GL}_n(\mathcal{O});\, H_*(M_n(\overline{R}))_{\mathrm{GL}_n(k)}\big)\\
&\quad \cong H_*(\mathrm{GL}(\mathcal{O})) \otimes H_*(M(\overline{R}))_{\mathrm{GL}(k)}.
\end{aligned}$$

PROOF OF (ii). Consider the diagram

$$\begin{array}{lccccc}
\text{(I)} & \mathrm{BG}(\overline{A}) & \to & \mathrm{BGL}(A) & \underset{s_1}{\overset{B_\varepsilon}{\rightleftarrows}} & \mathrm{BGL}(\mathcal{O}) \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{(II)} & F & \xrightarrow{i} & \mathrm{BGL}^+(A) & \underset{s_2}{\overset{B_\varepsilon +}{\rightleftarrows}} & \mathrm{BGL}^+(\mathcal{O}).
\end{array}$$

Though the plus-construction is not functorial, we may choose to perform it once and for all on $\mathrm{BGL}(\mathbf{Z}) \subset \mathrm{BGL}(\mathcal{O}) \subset \mathrm{BGL}(A)$, thus obtaining a commutative diagram. The section s_1 is induced by $\mathcal{O} \to A$, and s_2 is induced from s_1 by the universal property of the plus-construction. Since $\mathrm{BGL}^+(A)$ is an infinite loop space, we can use its multiplication μ to define a homotopy equivalence $\bar{\mu}$: $F \times \mathrm{BGL}^+(\mathcal{O}) \to \mathrm{BGL}(A)^+$, $\bar{\mu}(x, y) = \mu(ix, s_2 y)$. Because $\mathrm{BGL}^+(A) \to \mathrm{BGL}^+(\mathcal{O})$ is an infinite loop map $\bar{\mu}$ induces an equivalence of the fibration (II) with the trivial fibration (III) $F \to F \times \mathrm{BGL}^+\mathcal{O} \to \mathrm{BGL}^+(\mathcal{O})$. So we consider the diagram

$$\begin{array}{lccccc}
\text{(I)} & \mathrm{BG}(\overline{A}) & \to & \mathrm{BGL}(A) & \to & \mathrm{BGL}(\mathcal{O}) \\
 & \downarrow & & b\downarrow & & \downarrow \\
\text{(II)} & F & \to & \mathrm{BGL}^+(A) & \to & \mathrm{BGL}^+(\mathcal{O}) \\
 & \uparrow & & \bar{\mu}\uparrow & & \uparrow \\
\text{(III)} & F & \to & F \times \mathrm{BGL}^+(\mathcal{O}) & \to & \mathrm{BGL}^+(\mathcal{O}).
\end{array}$$

Now b and $\bar{\mu}$ induce homology isomorphisms between total spaces and, by the comparison theorem for spectral sequences [**13**], isomorphisms of the spectral sequences. Since the spectral sequence of (III) collapses, so does the spectral sequence of (I), as claimed. We find also that

$$H_*(M(\overline{R};\mathbf{Q}))_{\mathrm{GL}(k)} \cong H_*(F;\mathbf{Q}).$$

Now (iii) follows with a little work from the formula in (i), from the collapsing result (ii), and from Theorem 1 and Proposition 2 of [**5**] (see also [**6**]).

$$\begin{aligned}
\dim_{\mathbf{Q}} \operatorname{Prim} H_*(F) &= \dim_{\mathbf{Q}} \operatorname{Prim}\big(H_*(M(\overline{R}))_{\mathrm{Gl}(k)}\big)\\
&= \dim_{\mathbf{Q}} \operatorname{Prim}\big(H_*(M(\overline{R}))_{\mathfrak{gl}(k)}\big)\\
&= \dim_{\mathbf{Q}} \operatorname{Prim} H_*\big(R_{k/\mathbf{Q}}(M(\overline{R}));\mathbf{Q}\big)_{(R_{k/\mathbf{Q}}\mathfrak{gl})(\mathbf{Q})}\\
&= \dim_{\mathbf{C}} \operatorname{Prim} H_*\big(\mathbf{C}\otimes R_{k/\mathbf{Q}}(M(\overline{R}));\mathbf{C}\big)_{\mathbf{C}\otimes(R_{k/\mathbf{Q}}\mathfrak{gl})(\mathbf{Q})}\\
&= \dim_{\mathbf{C}} \operatorname{Prim} H_*\big(M(\mathbf{C}\otimes\overline{R})^d;\mathbf{C}\big)_{\mathfrak{gl}(\mathbf{C})^d}\\
&= d\cdot\dim_{\mathbf{C}} \operatorname{Prim} H_*\big(M(\mathbf{C}\otimes\overline{R});\mathbf{C}\big)_{\mathfrak{gl}(\mathbf{C})}\\
&= d\cdot\dim_{k} \operatorname{Prim} H_*\big(M(\overline{R});k\big)_{\mathfrak{gl}(k)}.
\end{aligned}$$

The results of [**5**] are that $\operatorname{Prim} H_*(M(\overline{R});k)_{\mathfrak{gl}(k)} \cong HC_{*-1}(\overline{R})$, so this ends the proof.

Notice that if $k=\mathbf{Q}$, this manipulation involving extending scalars and restricting again is superfluous, and we get a direct identification of $\operatorname{Prim} H_*(F)$ with $HC_{*-1}(\overline{R})$.

To prove Theorem 2 we first observe that $\mathrm{G}_n(\overline{A}) \hookrightarrow \mathrm{G}_n(\overline{R})$ ($\overline{R}=\mathbf{Q}\otimes\overline{A}$) induces isomorphisms in rational homology. We can show this considering for $m=1,2,\dots,N-1$ the maps of short exact sequences

$$\begin{array}{ccccc}
I_n+M_n(\overline{A}^m/\overline{A}^{m+1}) & \rightarrowtail & \mathrm{G}_n(\overline{A}/\overline{A}^{m+1}) & \twoheadrightarrow & \mathrm{G}_n(\overline{A}/\overline{A}^m)\\
\downarrow & & \downarrow & & \downarrow\\
I_n+M_n(\overline{R}^m/\overline{R}^{m+1}) & \rightarrowtail & \mathrm{G}_n(\overline{R}/\overline{R}^{m+1}) & \twoheadrightarrow & \mathrm{G}_n(\overline{R}/\overline{R}^m).
\end{array}$$

(Assume $\overline{A}^N=0$.) So we may replace $\mathrm{G}_n(\overline{A})$ by the $\mathbf{Q}$-homologically equivalent group $\mathrm{G}_n(\overline{R})$ which is torsion-free, nilpotent, and uniquely-divisible. To show that $M_n(\overline{R})$ is the nilpotent Lie algebra corresponding to $\mathrm{G}_n(\overline{R})$, we recall some of the general procedure. The nilpotent group G associated to a nilpotent Lie algebra $\mathfrak{g}$ may be realized as the underlying space of the algebra with the group law determined by the brackets via the Baker–Campbell–Hausdorff formula, which involves a formal exponentiation. In [**8**, Appendix A], the construction is treated in the following manner: Complete the universal enveloping algebra of $\mathfrak{g}$ with respect to its augmentation ideal, obtaining a complete Hopf algebra $\hat{\mathscr{U}}(\mathfrak{g})$. Now the exponential makes sense in $\hat{\mathscr{U}}(\mathfrak{g})$ and defines a one-to-one correspondence between $\mathfrak{g}\cong\operatorname{Prim}\hat{\mathscr{U}}(\mathfrak{g})$ and the group $\mathrm{G}\hat{\mathscr{U}}(\mathfrak{g})=\{x\in\hat{\mathscr{U}}(\mathfrak{g})\mid\Delta x=x\otimes x\}$, the group-like elements of $\hat{\mathscr{U}}(\mathfrak{g})$. Since $\mathrm{G}\hat{\mathscr{U}}(\mathfrak{g})\subset 1+\overline{\hat{\mathscr{U}}(\mathfrak{g})}$, the formal logarithm makes sense and is the inverse map. On the other hand, because $\overline{R}$ is nilpotent the matrix exponential $M_n(\bar{R}) \to M_n(R)$ makes sense and defines a one-to-one correspondence of $M_n(\bar{R})$ with the group of matrices $I + M_n(\bar{R}) = G_n(\bar{R}) \subset M_n(R)$. The inverse is, of course, the matrix logarithm. This ends the proof.

2. We close with a few remarks on computation of cyclic homology. Cyclic homology over a field k of characteristic zero is most quickly defined by taking the homology of a certain quotient complex of the standard Hochschild complex of a k-algebra S: $HC_l(S)$ is the lth homology of the complex which is, in degree l, $S^{\otimes l+1}$ modulo the subspace spanned by

$$\left\{ r_0 \otimes \cdots \otimes r_l - (-1)^l r_l \otimes r_0 \otimes \cdots \otimes r_{l-1} \right\}$$

and whose boundary operator is induced by the Hochschild boundary formula

$$b(r_0 \otimes \cdots \otimes r_l) = \sum_{i=0}^{l-1} (-1)^i r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_l$$
$$+ (-1)^l r_l r_0 \otimes r_1 \otimes \cdots \otimes r_{l-1}.$$

This is the indexing convention of **[6]** and it is one off from that of **[5]**. **[6]** also contains a formula for the cyclic homology of the augmentation ideal I of a ring of dual numbers $S = k \oplus I$ where $I^2 = 0$. Namely,

Fact 1.

$$HC_l(I) = I^{\otimes l+1} / \left(r_0 \otimes \cdots \otimes r_l - (-1)^l r_l \otimes r_0 \otimes \cdots \otimes r_{l-1} \right)$$

since the Hochschild boundary operator induces the zero map in this case. So, if I is one dimensional, $I = K \cdot \varepsilon$, we deduce that $HC_l(I)$ is zero if l is odd and one dimensional if l is even. So this fact and Theorem 1 reproduce Soulé's result **[9]**.

We will need the following general result relating the cyclic homology of an augmented k-algebra $S = k \oplus \bar{S}$ to the cyclic homology of its augmentation ideal $\bar{S}$, a ring without unit:

Fact 2.

$$HC_*(S) = HC_*(k) \oplus HC_*(\bar{S}).$$

Note that the direct computation gives that $HC_l(k)$ is a copy of k if l is even and zero if l is odd, so it suffices to compute $HC_*(S)$, if one wants $HC_*(\bar{S})$.

To get at $HC_*(S)$ we need the following general results relating Hochschild homology and cyclic homology (Theorem 1.9 of **[6]**).

Fact 3. *For any k-algebra S, there is a spectral sequence abutting to* $HC_*(S)$ *with* $E^1_{p,q} = H_{p-q}(S)$, *the* $(p-q)$*th Hochschild homology group of S.*

Pictorially,

$$E^1_{**} = \begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ H_3(S) & \leftarrow & H_2(S) & \leftarrow & H_1(S) & \leftarrow & H_0(S) \\ H_2(S) & \leftarrow & H_1(S) & \leftarrow & H_0(S) & & \\ H_1(S) & \overset{d^1}{\leftarrow} & H_0(S) & & & & \\ H_0(S) & & & & & & \end{array} \Rightarrow HC_*(S).$$

Now the complex $H_0(S) \xrightarrow{d^1} H_1(S) \xrightarrow{d^1} H_l(S) \xrightarrow{d^1} H_{l+1}(S) \to \cdots$ is conventionally called the deRham complex of S and its homology groups are called the deRham homology groups of S, $H_*^{\mathrm{dR}}(S)$. Refer to **[6]** for the explanation of this terminology. Now the result from **[14]** we need is Corollary II.4.3, which we state as

Fact 4 (*the homotopy invariance of deRham homology*). *A k-derivation D of S extends to an endomorphism of L_D of the deRham complex of S which therefore induces an endomorphism of L_D of $H_*^{\mathrm{dR}}(S)$. In fact, on $H_*^{\mathrm{dR}}(S)$, $L_D = 0$ if* char $k = 0$.

This leads to Goodwillie's computational result:

PROPOSITION 5. *Suppose S is a commutative k-algebra which admits a positive grading. Then the Hochschild, cyclic and deRham homology groups are all internally graded S modules and $H_p^{\mathrm{dR}}(S)_q = 0$ unless $p = q = 0$.*

PROOF. Work through the definitions in Chapter II, §4 of **[14]** in case $D: S \to S$ is given by $D(s) = (\deg s)s$: The endomorphism L_D acts by multiplication by q in $H_p(S)_q$ and $H_p^{\mathrm{dR}}(S)_q$. Now quote the preceding fact, notice that $H_p(S)_0 = 0$ if $p > 0$, and observe that multiplication by a nonzero scalar coincides with the zero map only on the zero vector space.

In case $S = k[T]/(T^{n+1})$ we can compute $H_*(S) = \mathrm{Tor}_*^{S \otimes S}(S, S)$ from a periodic resolution of the $S \otimes S$ module S (similar to the periodic resolution one uses to compute group homology $H_*(\mathbf{Z}/n, M)$) and one finds

$$\dim H_0(S) = n + 1, \qquad \dim H_l(S) = n \quad \text{for } l \geqslant 1.$$

The vanishing result then implies that $d^1: H_{2j}(S) \to H_{2j+1}(S)$ is onto for all $j \geqslant 0$ and $d^1: H_{2j-1}(S) \to H_{2j}(S)$ is zero for all $j \geqslant 1$. Thus the dimensions of the $E_{p,q}^2$ terms of the spectral sequence work out to

$$\dim E_{0,0}^2 = n + 1,$$

$$\dim E_{p,q}^2 = \begin{cases} n, & q > 0, \text{ even}; p = 0, \\ 1, & p = q > 0, \\ 0, & \text{otherwise}. \end{cases}$$

The diagonal corresponds to the summand $HC_*(k)$ in the splitting $HC_*(k) \oplus HC_*(\bar{S}) \cong HC_*(S)$, so one easily deduces

$$\dim HC_l(\bar{S}) = \begin{cases} n & \text{if } l \text{ is even}, \\ 0 & \text{if } l \text{ is odd}, \end{cases}$$

and we may plug this computation into our main theorem.

REFERENCES

1. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272.

2. W. G. Dwyer, W.-C. Hsiang and R. Staffeldt, *Pseudo-isotopy and invariant theory*, Topology **19** (1980), 367–385.

3. F. T. Farrell and W.-C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Algebraic and Geometric Topology, Part 1 (J. Milgram, ed.), Proc. Sympos. Pure Math., Vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 325–337.

4. P. A. Griffiths and J. W. Morgan, *Rational homotopy theory and differential forms*, Birkhäuser, Boston, Mass., 1981.

5. J.-L. Loday and D. G. Quillen, *Homologie cyclique et homologie de* 1*-algèbre de Lie des matrices*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 295–297.

6. ______, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

7. C. Procesi, *Invariant theory of* $n \times n$ *matrices*, Adv. in Math. **19** (1976), 306–381.

8. D. G. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.

9. C. Soulé, *Rational K-theory of the dual numbers of a ring of algebraic integers*, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 402–409.

10. W. van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), 269–295.

11. A. Weil, *Adeles and algebraic groups*, Birkhäuser, Boston, Mass., 1982.

12. H. Weyl, *Classical groups*, Princeton Univ. Press, Princeton, N. J., 1946.

13. E. C. Zeeman, *A proof of the comparison theorem for spectral sequences*, Proc. Cambridge Philos. Soc. **53** (1957), 57–62.

14. T. G. Goodwillie, *Cyclic homology and the free loop space*, Preprint.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802