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EXAMPLES ON HARMONIC MEASURE AND NORMAL NUMBERS[1]

JANG-MEI WU

ABSTRACT. Suppose that F is a bounded set in $\mathbf{R}^m$, $m \geq 2$, with positive capacity. Add to F a disjoint set E so that $E \cup F$ is closed, and let $D = \mathbf{R}^m \backslash (E \cup F)$. Under what conditions on the added set E do we have harmonic measure $\omega(F, D) = 0$? It turns out that besides the size of E near F, the location of E relative to F also plays an important role. Our example, based on normal numbers, stresses this fact.

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THEOREM. *Let D be a bounded domain in $\mathbf{R}^m$, $m \geq 2$, F be a subset of ∂D with $\Lambda^{m-1}(F) = 0$, and $E = \mathbf{R}^m \backslash (D \cup F)$. Suppose that F lies also on some quasi-smooth curve Γ when $m = 2$, on some* BMO_1 *surface Γ when $m \geq 3$. And suppose that at each $a \in F$, $0 < r < \frac{1}{4}$, there is a closed set $T \subseteq E \cap B(a, r)$ so that*

$$\text{capacity}(T) \geq \text{capacity}(B(0, c_1 r)) \tag{1}$$

and

$$\operatorname{dist}(T, F) > c_2 r; \tag{2}$$

also

$$\operatorname{diam}(T) < c_2 r/3 \quad \textit{when } m = 2, \tag{3}$$

where c_1 and c_2 are constants in $(0, 1)$. Then $\omega(F, D) = 0$.

By $B(a, r)$ we mean $\{x \in \mathbf{R}^m : |x - a| < r\}$; and by capacity we mean $(m-2)$-capacity if $m \geq 3$, and logarithmic capacity if $m = 2$. See [**2**, 1.XIII] for their properties.

When $m \geq 3$, (1) is equivalent to

$$\text{capacity}(T) \geq c \,\text{capacity}\, B(0, r). \tag{1'}$$

However, when $m = 2$, (1) is more restrictive than (1′).

The surface Γ and ∂D are in general distinct; no smoothness condition is imposed on ∂D. The problem is very different if ∂D is quasi-smooth or BMO_1 (see [**4**]). When $m \geq 3$, topological properties of D are less important in studying $\omega(F, D)$:

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there exist a topological ball D in $\mathbf{R}^3$, a set F lying on ∂D and on a plane, so that $\dim(F) = 1$, but $\omega(F, D) > 0$ [**9**].

In the theorem, we have three conditions on F: (a) it lies on a surface Γ with minimum smoothness, (b) $\mathbf{R}^m \backslash (D \cup F)$ is big near each point in F in the capacity sense, and (c) $\mathbf{R}^m \backslash (D \cup F)$ is untangled from F as in (2). Γ cannot be too general, the theorem is not true if Γ is a quasi-circle (see [**7**]) and it is quite clear that (1) in some sense is necessary. We shall show in Example 3 that if (a) and (b) are satisfied, but the part of $\mathbf{R}^m \backslash (D \cup F)$ that is big near F is not separated from F, we may still have $\Lambda^{m-1}(F) = 0$ but $\omega(F) > 0$.

The proof shows that the theorem still holds when F lies on a slightly more general surface, namely,

(4) Γ is a topological sphere in $\mathbf{R}^m$, whose interior Ω_1 and exterior Ω_2 are both nontangentially accessible domains, and on Γ, $\Lambda^{m-1}(E) = 0 \Rightarrow \omega(E, \Omega_i) = 0$ for $i = 1, 2$.

See [**4**] for the definitions of quasi-smooth curves (also called chord-arc), BMO_1 surfaces and nontangentially accessible domains, and their relations.

This theorem is an improvement of the one in [**9**]. There $\mathbf{R}^m \backslash D$ satisfies a corkscrew condition; that is, T can be chosen to be a ball in $\mathbf{R}^m$ with

$$\text{radius } T > cr. \tag{5}$$

It is also closely related to Theorem 2 in [**8**] and Theorem 3 in [**7**] when $m = 2$ and D is simply-connected. However, the present theorem does not imply the results in [**7** or **8**], because, for a set F lying on the boundary of a simply-connected domain, we may not be able to find T so that the conditions (1) and (2) are both fulfilled.

To prove the theorem, we assume that Γ satisfies the more general condition (4) and follows the steps in [**9**]. Because our assumption on the size of T is in terms of capacity, which is weaker than (5), we need a replacement of (5) in terms of harmonic measure, which is sufficient for the proof of the theorem. First, by the countable subadditivity of capacity for $m \geq 3$ and by (3) for $m = 2$, we can find c_3, c_4 depending on m, c_1, and c_2 only so that, for any $a \in F$, $0 < r < \frac{1}{4}$, there exists $P \in B(a, r)$ with $\operatorname{dist}(P, F) > 3c_3 r$ and

$$\operatorname{capacity}(B(P, c_3 r) \cap T) > \operatorname{capacity}(B(a, c_4 r)).$$

Therefore, for $|Q - P| = 2c_3 r$,

$$w^Q(T \cap \overline{B(P, c_3 r)}, B(P, 3c_3 r) \backslash (T \cap \overline{B(P, c_3 r)})) > c_5 > 0.$$

We remark that when $m = 2$, these statements do not follow from (1$'$). The rest of the proof follows the similar lines as in [**9**].

Here are some applications.

EXAMPLE 1. In $\mathbf{R}^m$, there exist disjoint sets E and F in $\overline{B(0, \sqrt{m})}$, whose union $E \cup F$ is closed, of zero Λ^{m-1}-measure, and each of which has positive capacity. However, $\omega(E, D) > 0, \omega(F, D) = 0$ where $D = B(0, m) \backslash (E \cup F)$. Moreover, for any β, α in $[m-2, m-1]$, E and F can be chosen with $\dim E = \beta$ and $\dim F = \alpha$; in particular, it is possible to have $\dim E = m-2$, $\dim F = m-1$.

We give the construction for $m = 3$; some necessary changes are needed for other m's. Given a sequence $\{r_n\}_1^\infty \supseteq (0, \frac{1}{2})$, we construct the corresponding Cantor set

S: Let $S_{0,1} = [0,1]$; $S_{1,1}$ and $S_{1,2}$ be the two closed intervals of length r_1 each, left after the middle $(1-2r_1)$ portion of $[0,1]$ is removed. At the nth step, let $\{S_{n,k}\}_{k=1}^{2^n}$ be the closed intervals of length $\prod_1^n r_j$ each, left after the middle $(1-2r_n)$ portion of each $\{S_{n-1,k}\}_1^{2^{n-1}}$ is removed. Let

$$S = \bigcap_1^\infty \bigcup_{k=1}^{2^n} S_{n,k}.$$

Let $\{A_{n,l}\}_{l=1}^{4^n} = \{S_{n,k} \times S_{n,j} : 1 \le k, j \le 2^n\}$ and $A = \bigcap_{n=1}^\infty \left(\bigcup_{l=1}^{4^n} A_{n,l}\right)$ the Cantor set on $[0,1] \times [0,1]$. We choose

$$r_n = \begin{cases} \dfrac{1}{4}\left(\dfrac{n+3}{n+2}\right)^2, & \alpha = 1, \\ 4^{-1/\alpha}, & 1 < \alpha < 2, \\ \dfrac{1}{2}\dfrac{n}{n+1}, & \alpha = 2. \end{cases} \tag{6}$$

Standard calculation shows that $\dim A = \alpha$, $\Lambda^2(A) = 0$ and 1-capacity$(A) \equiv c_\alpha > 0$. We let F be $A \times \{0\}$ in R^3.

Following the construction of A, we may construct a Cantor set $E_{n,l}$ in $A_{n,l} \times \{\prod_1^n r_j\} \subseteq \mathbf{R}^2 \times \{\prod_1^n r_j\} \subseteq \mathbf{R}^3$ satisfying $\dim E_{n,l} = \beta$, $\Lambda^2(E_{n,l}) = 0$ and

$$\text{capacity}(E_{n,l}) = c_\beta \times \prod_1^n r_j. \tag{7}$$

Let $E = \bigcup_{n=1}^\infty \left(\bigcup_{l=1}^{4^n} E_{n,l}\right)$, thus $\dim E = \beta$, $\Lambda^2(E) = 0$ and capacity$(E) > 0$. Let $D = B(0,m) \backslash (E \cup F)$. Because of (7), we see that, at each $a \in F$, for $\sqrt{m}\prod_1^n r_j \le r < \sqrt{m}\prod_1^{n-1} r_j$, there exists some $E_{n,l}$ so that

$$\text{dist}(E_{n,l}, A) > cr \quad \text{and} \quad \text{capacity}(E_{n,l}) > cr.$$

From the theorem, it follows that $\omega(F, D) = 0$. Since

$$\text{capacity}(E \cup F) > 0,$$

we must have $\omega(E, D) > 0$.

EXAMPLE 2. Let

$$r_n = \begin{cases} (n+2)^{-1}, & \alpha = 0, \\ 2^{-1/\alpha}, & 0 < \alpha < 1, \\ n/2(n+1), & \alpha = 1. \end{cases}$$

Let α_1 and α_2 be numbers in $[0,1]$, not both zero, and S_j, $j = 1, 2$, be the Cantor set on $[0,1]$ corresponding to $\{r_n\}$ with $\alpha = \alpha_j$. Clearly, $\dim S_j = \alpha_j$. Let $D = \mathbf{R}^2 \backslash (S_1 \times S_2)$ be the domain in $\mathbf{R}^2$, whose boundary is the planar Cantor set $S_1 \times S_2$ of dimension $\alpha_1 + \alpha_2$. Then any line parallel to either coordinate axis meets ∂D on a set of zero harmonic measure and of dimension α_1 or α_2. The example is particularly interesting when $\alpha_1 = 0$ (or $\alpha_2 = 0$).

Some complementary examples on domains whose boundaries consist of other Cantor sets can be found in [**8**].

The next example shows that condition (2)—ensuring that E and F are untangled—is not superfluous.

The author thanks R. Kaufman for bringing the concept of normal numbers to her attention.

EXAMPLE 3. In $\mathbf{R}^m$, there exist disjoint sets E and F in $B(0,\sqrt{m})\cap(R^{m-1}\times\{0\})$, with the properties that each set has zero Λ^{m-1}-measure, positive capacity, that $E\cup F$ is closed, and that, for any $a\in F$, $0<r<\frac{1}{4}$,

$$\text{capacity}(B(a,r)\cap E) > \text{ capacity } B(0,cr). \tag{8}$$

However, $w(F,D)>0$, where $D=B(0,m)\backslash(E\cup F)$. Moreover, for any $\alpha\in(m-2,m-1]$, we can choose E and F so that $\dim(E\cup F)=\alpha$.

We first consider the case $m=3$ and $1<\alpha\leq 2$, and let r_n be as in (6), S and A be the Cantor sets on $[0,1]$ or on $[0,1]\times[0,1]$ as defined in Example 1, and let $D=B(0,3)\backslash A$. We identify $\mathbf{R}^2$ with $\mathbf{R}^2\times\{0\}$.

We represent a number $a\in S$ by $(a_1,a_2,a_3,\ldots)$, where $a_n=0$ if a is in the left half of some interval $S_{n-1,k}$, and $a_n=1$ if a is in the right half. We define $f\colon S\to[0,1]$ by

$$f(a)=\sum_1^\infty\frac{a_n}{2^n}.$$

A number a in $[0,1]$ is called *simply normal in the scale of* 2 if its binary expansion $\sum_1^\infty(a_n/2^n)$ has the property that

$$n^{-1}\sum_{j=1}^n a_j\to\frac{1}{2}\qquad\text{as } n\to\infty.$$

It is known [**1** and **3**, p. 124] that Lebesgue-almost every point in $[0,1]$ is simply normal in the scale of 2. We let

$$N=\{a\in S\colon f(a)\text{ is simply normal in the scale of }2\},$$

β be a number in $(0,\frac{1}{2})$ which satisfies

$$1-\beta<2^{-1/\alpha} \tag{9}$$

and $M=\{a\in S\colon n^{-1}\sum_1^r a_j\to\beta,\text{ as } n\to\infty\}$.

Because $N\times N$ has Lebesgue measure 1,

$$\text{capacity}(N\times N)>0. \tag{10}$$

We claim that

$$\text{capacity}(M\times M)>0. \tag{11}$$

Assuming this is true, we proceed as follows. Because r_n are constant for $1<\alpha<2$ and r_n are increasing for $\alpha=2$, by an appropriate scaling, we obtain, for any $a\in A$, $0<r<1$,

$$\text{capacity}(N\times N\cap B(a,r))>cr, \tag{12}$$

$$\text{capacity}(M\times M\cap B(a,r))>cr. \tag{13}$$

Since capacity$(A) > 0$, we have $\omega(A, D) > 0$. Therefore, at least one of the three sets, $N \times N$, $M \times M$, or $A \backslash (N \times N \cup M \times M)$ has positive harmonic measure with respect to D.

Question. Which one?

We choose F to be any one of these three which has positive harmonic measure, and $E = A \backslash F$. Then (8) follows from (12) or (13).

To prove (11), we define a measure λ on $[0,1]$ by

$$\lambda(I) = \prod_{j=1}^{n} [(2\beta - 1)a_j + 1 - \beta] \tag{14}$$

if I is the closed interval

$$\left[\sum_{1}^{n} \frac{a_j}{2^j}, \frac{1}{2^n} + \sum_{1}^{n} \frac{a_j}{2^j}\right],$$

for any $n > 0$ and $a_j = 0$ or 1, and let ν be the induced measure on S, $\nu(E) = \lambda(f(E))$.

It is clear that $\nu([0,1]) = 1$.

We need to show that

$$\nu(M) = 1. \tag{15}$$

Let X_j be random variables on $([0,1], \lambda)$ so that $X_j(a) = a_j$. They are independent, uniformly bounded and $E(X_j) = \beta$. It is proved in [**6**] by using ideas from [**5**, p. 131] that

$$\sum_{1}^{n} X_j - n\beta = o(n^{2/3}), \qquad n \to \infty, \tag{16}$$

for λ-almost every a in $[0,1]$. This says that $\lambda(f(M)) = 1$ or $\nu(M) = 1$.

Because of (14),

$$\nu \times \nu \left(M \times M \cap B\left(P, \prod_{j=1}^{n} r_j\right)\right) \leq 4(1-\beta)^{2n}$$

for any $P \in \mathbf{R}^3$. With the aid of (6) and (9) we conclude

$$\begin{aligned} &\int_{M \times M} |P - Q|^{-1}\, d\nu \times \nu(Q) \\ &\quad \leq C\left[\frac{1}{1 - 2r_1} + \sum_{n=1}^{\infty} \frac{4}{\left(\prod_{j=1}^{n} r_j\right)(1 - 2r_{n+1})}(1-\beta)^{2n}\right] < C_{\alpha,\beta} < +\infty \end{aligned}$$

for any $P \in \mathbf{R}^3$. This, together with (15), shows that $M \times M$ has positive capacity.

(10) can also be obtained by replacing β by $\frac{1}{2}$ in the proof above . This completes the case $m = 3$. To arrive at the example for $m \geq 4$, we need only some routine changes.

An alternative way to show (11) is calculating the Hausdorff dimension of $M \times M$. This approach is particularly convenient for $m = 2$, when the logarithmic potential is harder to estimate.

When $m = 2,\ 0 < \alpha \leq 1$, we let

$$r_n = \begin{cases} 2^{-1/\alpha}, & 0 < \alpha < 1, \\ n/2(n+1), & \alpha = 1, \end{cases}$$

and construct the Cantor set S of dimension α, M and N, accordingly. Combining results in [**1** and **6**], we see that

$$\dim f(M) = \frac{\beta \log \beta + (1-\beta)\log(1-\beta)}{\log \frac{1}{2}}.$$

By translating coverings of M through f, we can show that

$$\dim M = \alpha[\beta \log \beta + (1-\beta)\log(1-\beta)]/\log \tfrac{1}{2} > 0.$$

Since $f(N)$ are the simply normal numbers, $\dim f(N) = 1$ and $\dim N = \alpha$. Now let $A \equiv S$, and F be any one of the three sets N, M and $A \backslash (N \cup M)$ which has positive harmonic measure with respect to $D \equiv B(0,2)\backslash A$. The properties in the example are satisfied.

Question. We do not know whether A can be chosen in Example 3 to have $\dim A = m - 2$. When $m = 3$, $\alpha = 1$, $\{r_n\}$ is not increasing, thus (12) does not follow from (10) automatically, and we do not even know whether r_n can be chosen so that capacity$(A \cap B(a,r)) > cr$ for all $a \in A$, $0 < r < \frac{1}{4}$.

REMARK. When $m \geq 3$, the domain D in Example 3 can be made into a topological ball, by deleting a branching tree T, connecting $B(0,m)$ to A, from $B(0,m)\backslash A$. This is possible because line segments have capacity zero when $m \geq 3$. The branches of T should be chosen carefully so that the ratio of capacity(T) to capacity(A) is sufficiently small.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, ILLINOIS 61801