PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 2, October 1985

EXPLICIT EXAMPLES OF BLOCH FUNCTIONS IN EVERY H^p SPACE, BUT NOT IN BMOA

FINBARR HOLLAND AND J. BRIAN TWOMEY

ABSTRACT. It is shown how to construct analytic functions, with nonnegative Taylor coefficients, that belong to the intersection of the space of Bloch functions and all the H^p spaces and yet do not have bounded mean oscillation.

1. Introduction. A function f, analytic on the open unit disc D, is said to belong to the Bloch space B if

$$\sup\{(1-|z|^2)|f'(z)|: z \in D\} < \infty,$$

and to be in BMOA if it is in the Hardy space H^2 and its radial limit function has *bounded mean oscillation* on the unit circle.

Both B and BMOA are dual spaces of Banach spaces of analytic functions: B is (isomorphic to) the dual of the space I of functions g analytic on D such that

$$\iint_D |g'(z)|\, dx\, dy < \infty,$$

while BMOA is (isomorphic to) the dual of the Hardy space H^1. The former statement was established in [**1**], the latter in [**5**].

Evidently, $I \subset H^1$ and, for $p > 1$, $H^p \subset H^1$; the inclusions are continuous. Hence, by duality, BMOA$\subset B$ and BMOA$\subset H^q$, for all $q > 1$. Thus the containment relation

$$\text{BMOA} \subset B \cap \{H^p : p > 0\}$$

follows. The question arises: Is the inclusion strict?

Using arguments involving the existence of a universal covering map and stability properties [**9**] of BMOA, it was shown in [**4**] how to create functions in the complement $B \cap \{H^p : p > 0\} \backslash \text{BMOA}$. In the same paper, the authors called for more explicit examples of such functions. The present note is a response to that request.

Elsewhere [**6**], the need arose to exhibit non-BMOA functions with nonnegative Taylor coefficients in $B \cap H^2$. The technique we use here to construct such functions in the smaller space $B \cap \{H^p : p > 0\}$ is an elaboration of the one outlined in [**6**].

2. Criteria for membership of B, BMOA and H^p. Our method for constructing functions of the desired kind is based on criteria for membership of the three spaces B, BMOA and H^p, which, for convenience, we recall here:

Received by the editors December 11, 1984.
1980 *Mathematics Subject Classification*. Primary 30D55, 30D60; Secondary 30B10.
Key words and phrases. Bloch functions, H^p spaces, bounded mean oscillation, dual spaces.

Let $a_n \geq 0$, for $n = 0, 1, \dots$, and $f(z) = \sum a_n z^n$ $(z \in D)$. It is an easy consequence of the definition that $f \in B$ if and only if

$$\sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} k a_k < \infty,$$

equivalently, if and only if

$$\sup_{k \geq 0} \sum_{2^k}^{2^{k+1}} a_j < \infty.$$

The classical Hausdorff-Young theorem [**7**] tells us that f belongs to H^p, for a fixed $p \geq 2$, if $\sum (a_n)^q < \infty$, where $q = p/(p-1)$ is the conjugate of p. In particular, then, f belongs to H^p for all $p > 0$ if the last displayed series is convergent for every $q > 1$.

C. Fefferman has given a necessary and sufficient condition for an analytic function, having nonnegative Taylor coefficients, to belong to BMOA [**2**], and we exploit that here. Proofs of this criterion can be found in [**3**, **6** and **8**]. It goes as follows: f belongs to BMOA if and only if

$$\sup_{n \geq 1} \sum_{j=1}^{\infty} \left(\sum_{r=0}^{n-1} a_{jn+r} \right)^2 < \infty.$$

3. The construction. Let $F_0 = \{0\}$. For $k > 0$, let $m(k)$ denote the integer part of $2^{\sqrt{k}} - 2^{\sqrt{(k-1)}}$, and set $F_k = \{2^k + j : j = 0, 1, \dots, m(k)\}$.

Let $E_0 = \{1\}$ and for $k > 0$ let $E_k = \{2^k + j : j \in \bigcup \{F_i : 0 \leq i \leq k-1\}\}$. Then $E_k \subset [2^k, 2^{k+1})$, and $n(k)$, the number of elements in E_k, is equal to $1 + \sum_{j=1}^{k-1} [m(j) + 1]$ and therefore satisfies the inequalities

$$2^{\sqrt{(k-1)}} < n(k) \leq k + 2^{\sqrt{(k-1)}} \quad \text{for } k = 1, 2, \dots.$$

Define the sequence (a_n) as follows:

$$\begin{aligned} a_n &= \varepsilon_k = 2^{-\sqrt{k}} && \text{if } n \in E_k,\ k = 0, 1, \dots, \\ &= 0 && \text{if } n \notin E = \bigcup \{E_i : i \geq 0\}. \end{aligned}$$

Claim. The function f given by

$$f(z) = \sum a_n z^n = \sum_{k \geq 0} \varepsilon_k \sum_{n \in E_k} z^n$$

lies outside BMOA, but inside B and $\bigcap \{H^p : p > 0\}$.

First, $f \in B$. This follows from the fact that $n(k)$, the number of elements of E in any interval of the form $[2^k, 2^{k+1})$, is at most $3 \cdot 2^{\sqrt{k}}$ and $a_n = 2^{-\sqrt{k}}$ on E_k. Thus

$$\sum_{n=2^k}^{2^{k+1}-1} a_n \leq 2^{-\sqrt{k}} n(k) \leq 3 \quad \text{for } k = 0, 1, \dots.$$

Second, $f \in H^p$ for all $p > 0$. For, if $q > 1$, then

$$\sum (a_n)^q = \sum_{k=0}^{\infty} \left(\sum_{n \in E_k} (a_n)^q \right) = \sum_{k=0}^{\infty} (\varepsilon_k)^q n(k)$$

$$\leq 3 \cdot \sum_{k=0}^{\infty} 2^{-(q-1)\sqrt{k}} < \infty.$$

Third, $f \notin$ BMOA. To see this, note that the number of elements in E that belong to any interval of the form $[2^{s+m}, 2^{s+m} + 2^m - 1]$ is at least equal to $n(m)$. Hence, with $n = 2^m$,

$$\sum_{j=1}^{\infty} \left(\sum_{r=jn}^{jn+n-1} a_r \right)^2 \geq \sum_{s=0}^{\infty} \left(\sum_{r=2^{s+m}}^{2^{s+m}+2^m-1} a_r \right)^2$$

$$\geq \sum_{s=0}^{\infty} (n(m)\varepsilon_{s+m})^2 = (n(m))^2 \sum_{s=m}^{\infty} \varepsilon_s^2$$

$$\geq (n(m))^2 \int_m^{\infty} 2^{-2\sqrt{x}}\, dx \geq 2^{2\sqrt{(m-1)}} \sqrt{m} 2^{-2\sqrt{m}} \to \infty,$$

as m tends to infinity.

Thus the function f has all the desired properties.

REFERENCES

1. J. M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37. MR **50** #13536.
2. J. M. Anderson and A. L. Shields, *Coefficient multipliers of Bloch functions*, Trans. Amer. Math. Soc. **224** (1976), 255–265. MR **54** #7787.
3. F. F. Bonsall, *Boundedness of Hankel matrices*, J. London Math. Soc. (2) **29** (1984), 289–300.
4. Douglas Campbell, Joseph Cima and Kenneth Stephenson, *A Bloch function in all H^p classes, but not in* BMOA, Proc. Amer. Math. Soc. **78** (1980), 229–230. MR **80k**:30033.
5. C. L. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193. MR **56** #6263.
6. Finbarr Holland and David Walsh, *Boundedness criteria for Hankel operators*, Proc. Royal Irish Acad. **84A** (1984), 141–154.
7. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York-London-Sydney, 1968. MR **40** #1734.
8. W. T. Sledd and D. A. Steganga, *An H^1 multiplier theorem*, Ark. Mat. **19** (1981), 265–270. MR **84j**:42018.
9. Kenneth Stephenson, *Weak subordination and stable classes of meromorphic functions*, Trans. Amer. Math. Soc. **262** (1980), 565–577. MR **81m**:30029.

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND