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ON THE CLASSES ΛBV AND $V[\nu]$

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ABSTRACT. We prove inclusion relations between Waterman's and Chanturiya's classes and point to some corollaries thereof. The situation which occurs in connection with Zygmund's theorem for Waterman's classes is clarified.

1. Introduction. Let f be a real function of period 2π. Let Λ denote a nondecreasing sequence of positive numbers λ_n such that $\Sigma 1/\lambda_n$ diverges, $\{I_n\}$ a sequence of nonoverlapping intervals $I_n = [a_n, b_n] \subset [0, 2\pi]$ and let $f(I_n) = f(b_n) - f(a_n)$. In [**17**] D. Waterman has introduced the following concept of generalized bounded variation.

DEFINITION 1. A function f is said to be of Λ-bounded variation ($f \in \Lambda$BV) if for every choice of $\{I_n\}$ we have

$$\sum_{n=1}^{\infty} |f(I_n)|/\lambda_n < \infty.$$

The supremum of these sums is called the Λ-variation of f. For $\Lambda = \{n\}$ we say that f is of harmonic bounded variation ($f \in$ HBV).

This notion has its genesis in the joint work of C. Goffman and D. Waterman [**10**] on everywhere convergence of Fourier series for every change of variable. Relations to other generalizations of bounded variation, properties of functions of the class ΛBV (and those of Λ-variation function) as well as the convergence and summability properties of their Fourier series have been investigated in [**12, 13, 16–20**].

S. Perlman [**12**] has proved that the intersection of ΛBVs, taken over all sequences Λ, is the class BV of functions of bounded variation and the union of ΛBVs is the class of regulated functions. With respect to the problems of convergence of Fourier series, the class HBV appears to be of special importance [**17, 20**]. Perhaps the highest achievement are the definitive results for localization by square and rectangular sums for the Fourier series of functions of two variables obtained in [**11**], after a suitable definition of ΛBV in this case.

On the other hand, for everywhere bounded 2π-periodic functions, Z. A. Chanturiya [**3**] has introduced the concept of the modulus of variation.

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DEFINITION 2. The modulus of variation of a function f is the function $\nu_f(n)$ with domain the positive integers, defined by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|$$

where Π_n is an arbitrary system of n disjoint intervals $I_k \subset (0, 2\pi)$.

The modulus of variation of any function is nondecreasing and upwards convex. Functions of an integral argument with such properties are said to be moduli of variation. If the modulus of variation $\nu(n)$ is given, then $V[\nu]$ denotes the class of functions for which $\nu_f(n) = O(\nu(n))$ when $n \to \infty$.

By these means, in **[3–8]** Chanturiya has investigated uniform and absolute convergence of Fourier series, the latter in regard to the possibility of extension of the Zygmund and Bochkarev theorems [**21**, p. 241; **2**].

The purpose of this note is to contribute to unifying results.

2. Results. In what follows we suppose $\lambda_n \nearrow \infty$, since it is the case of interest.

THEOREM 1. $\Lambda\mathrm{BV} \subset V[n/(\sum_{i=1}^{n} 1/\lambda_i)]$.

PROOF. Let us take an arbitrary $f \in \Lambda\mathrm{BV}$ and fix it. From the definition of Λ-bounded variation follows the existence of the constant M such that for any system $\{I_k\}$ of n $(n = 1, 2, \ldots)$ disjoint subintervals of the interval $(0, 2\pi)$ we have

$$\begin{gathered}
|f(I_1)|/\lambda_1 + |f(I_2)|/\lambda_2 + \cdots + |f(I_n)|/\lambda_n \le M \\
|f(I_1)|/\lambda_2 + |f(I_2)|/\lambda_3 + \cdots + |f(I_n)|/\lambda_1 \le M \\
\vdots \\
|f(I_1)|/\lambda_n + |f(I_2)|/\lambda_1 + \cdots + |f(I_n)|/\lambda_{n-1} \le M.
\end{gathered}$$

By summation we get

$$\left(\sum_{i=1}^{n} 1/\lambda_i\right)\left(\sum_{k=1}^{n} |f(I_k)|\right) \le M$$

and the conclusion of the theorem follows.

By [**4**, Theorem 1], for any bounded function f,

$$\omega_1(\delta, f) = O\big(\delta \nu_f([1/\delta])\big) \qquad (\delta \to 0+),$$

where $\omega_1(\delta, f)$ is the integral modulus of continuity of f and $[1/\delta]$ is the largest integer less than or equal to $1/\delta$. From this and Theorem 1, we obtain

COROLLARY ([**14**, THEOREM 1; **16**, THEOREM 2]). *For $f \in \Lambda\mathrm{BV}$,*

$$\omega_1(\delta, f) = O\left(1 / \sum_{i=1}^{[1/\delta]} 1/\lambda_i\right)$$

and, therefore the Fourier coefficients of f are $O(1/\sum_{i=1}^{n} 1/\lambda_i)$.

THEOREM 2. ΛBV *contains every class* $V[\nu]$ *such that the condition*

$$\sum_{k=1}^{\infty} \Delta(1/\lambda_k)\nu(k) < \infty$$

is satisfied, where $\Delta a_k = a_k - a_{k+1}$.

PROOF. Let $\{I_k\}$, $k = 1,\dots,n$, be an arbitrary collection of nonoverlapping intervals, $I_k \subset [0, 2\pi]$. By partial summation we obtain

$$\begin{aligned}\sum_{k=1}^{n} |f(I_k)|/\lambda_k &= \sum_{k=1}^{n-1} \Delta(1/\lambda_k) \sum_{i=1}^{k} |f(I_i)| + 1/\lambda_n \sum_{i=1}^{n} |f(I_i)| \\ &\leq \sum_{k=1}^{n-1} \Delta(1/\lambda_k)\nu(k) + \nu(n)/\lambda_n\end{aligned}$$

and $\nu(n)/\lambda_n \leq \Sigma_{k=n}\Delta(1/\lambda_k)\nu(k)$.

Theorems 1 and 2 imply

THEOREM 3. (i) *If* $p > 1$ *and* $k\Delta(1/\lambda_k) = O(1)$, *then*

$$V\left[\frac{n}{(\Sigma_{i=1}^{n} 1/\lambda_i)^p}\right] \subset \Lambda\mathrm{BV};$$

(ii) $V[n^\alpha] \subset \{n^\beta\}\mathrm{BV} \subset V[n^\beta]$ *for* $0 < \alpha < \beta < 1$, $V[n/\ln^p n] \subset \mathrm{HBV} \subset V[n/\ln n]$;

(iii) ΛBV *contains all continuous functions f with modulus of continuity satisfying the condition*

$$\sum_{k=1}^{\infty} \Delta(1/\lambda_k) k\omega(1/k) < \infty;$$

(iv) Lip $\alpha \subset \{n^\beta\}$BV *for* $\alpha > 1 - \beta$. HBV *contains every Lipschitz class.*

PROOF. (i) Let us denote $u_k = k\Delta(1/\lambda_k)$, $S_n = \Sigma_{k=1}^{n} u_k$. By [**1**, Theorem 2, p. 21 (English transl., p. 6)], $S_n \to \infty$ $(n \to \infty)$. Hence

$$\sum_{k=1}^{\infty} \Delta(1/\lambda_k) \frac{k}{(\Sigma_{i=1}^{k} 1/\lambda_i)^p} \leq \sum_{k=1} \frac{u_k}{S_k^p} < \infty$$

[**1**, p. 905, Corollary to Theorem 1]. The conclusion follows then by Theorem 2.

(ii) Immediately from (i) and Theorem 1. For $V[n^\alpha]$ it is enough to take $p = (1-\alpha)/(1-\beta) > 1$.

(iii) Follows from Theorem 2 and the fact that $\nu_f(k) = O(k\omega(1/k, f))$ [**3**, Theorem 4].

(iv) Immediately from (iii).

3. Comments. (1) As an example of an application of Λ-bounded variation to absolute convergence of Fourier series, in the final section of [**17**] D. Waterman has generalized the theorem of Zygmund [**21**, p. 241]. As the reviewer has already remarked, this generalization loses its worth if $\lambda_n \geq n^{1/2}$. The same is true for a recent theorem of S. Wang [**16**, Theorem 7]. However, in case $\{n^\alpha\}BV$, $\alpha < 1/2$, a better result is possible. (See [**16**, Theorem 8]. The proof follows the line of the proof of Theorem 2 in [**4**] and the result is a consequence of that theorem, in view of our Theorem 1. As the matter of fact, both are equivalent, by Theorem 3(ii).) S. Wang remarks that no matter how the hypothesis $f \in \{n^\beta\}BV$ may be fulfilled with $\beta > 1/2$, the condition $f \in \{n^\beta\}BV$ cannot guarantee the absolute convergence of the Fourier series of $f \in \mathrm{Lip}\,\alpha$, if $\alpha \leq 1/2$. (If $\alpha > 1/2$ we have absolute convergence by the theorem of Bernstein, no other hypothesis being needed. See [**21**, p. 241].) The situation is the same with Chanturiya's classes $V[n^\beta]$, $\beta \geq 1/2$. We provide a rather obvious reason. If V_p denotes Wiener's class of p-bounded variation [**1**, p. 287], then $\mathrm{Lip}\,1/2 \subset V_2$ is immediate and $V_2 \subset V[n^{1/2}]$ follows by the Hölder inequality. Therefore, $\{n^\beta\}BV \cap \mathrm{Lip}\,\alpha \supset \mathrm{Lip}\,1/2$ for $\alpha \leq 1/2 < \beta$ and $\mathrm{Lip}\,1/2$, as is well known, contains a function whose Fourier series does not converge absolutely. More generally, given a modulus of continuity $\omega(\delta)$, if we denote by H^ω the class of functions $f \in C(0, 2\pi)$ with $\omega(\delta, f) = O(\omega(\delta))$ when $\delta \to 0$, we see from Theorem 3(iii) that $\{n^\beta\}BV$ contains every class H^ω with $\sum_{n=1}^\infty n^{-\beta}\omega(1/n) < \infty$. If $\sum_{n=1}^\infty n^{-1/2}\omega(1/n) = \infty$, then within the class $H^\omega \cap V[n^\alpha]$, $\alpha \geq 1/2$, there necessarily exists a function whose Fourier series is not absolutely convergent [**6**, p. 238]. The following extension by Chanturiya of the theorems of Zygmund and Bochkarev may be viewed as final in a certain sense.

THEOREM A [**7**, THEOREM 3]. *For all Fourier series of class $H^\omega \cap V[n^\alpha]$, $0 < \alpha < 1/2$, to be absolutely convergent, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} 1/n\{\omega(1/n)\}^{(1-2\alpha)/2(1-\alpha)} < \infty.$$

(2) The fact that $\{n^\alpha\}BV$ contains Wiener's class V_p (and hence $\mathrm{Lip}\,1/p$) for $\alpha > 1 - 1/p$ may be deduced also from Waterman's considerations of the relationships between $\varnothing$-bounded and Λ-bounded variation, in [**18**]. According to what is said in (1), no better extension of Zygmund's theorem for those classes than this given by Theorem A is possible. (Compare this to [**15**, Theorem 1]. See also [**4**, Theorem 3 and **15**, Theorem 2].)

(3) In view of S. Perlman's result mentioned in the introduction, the condition $\sum_{k=1}^{n}\rho_k = o(\ln n)$, where ρ_k is the absolute value of the kth Fourier coefficient, is sufficient for the continuity of function $f \in \Lambda BV$ (except for the removable discontinuities) by the well-known theorem of Lukâcs [**21**, p. 60]. In the case $\{n^\alpha\}BV$, $\alpha < 1/2$, the necessity can also be shown, in just the same way as Theorem 2 in [**4**] is proved. For $V[n^\alpha]$, $\alpha \geq 1/2$, this is no longer true as the Hardy-Littlewood's example

$$\sum_{n=1}^{\infty} n^{-1}e^{in\ln n}e^{inx} \in \mathrm{Lip}\,1/2$$

shows. Thus we get an extension of the theorem of N. Wiener [**1**, pp. 205–208; **21**, p. 40]. See [**8** and **9**] for somewhat more general statements.

(4) Finally, it is also interesting to compare conditions proposed by Chanturiya and Waterman so that Fourier series of all continuous functions of a certain class converge uniformly. Actually the result in **[17]** includes the corresponding one in **[5]**, by our second theorem.

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