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PERTURBATIONS AND GROUND STATES OF C^*-DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we show that if a C^*-dynamical system has an irreducible covariant representation, every relatively bounded $*$-derivation for its generator is also implemented by a relatively bounded selfadjoint operator for that associated with the dynamics. As its application, we assert that the existence of ground states of a C^*-dynamical system is stable under sufficiently small perturbations.

C^*-dynamical systems have been studied as a mathematical formulation for quantum mechanics, in particular, quantum statistical mechanics with time evolution. One of the most interesting problems is that concerned with perturbations. Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system, δ_α be its (infinitesimal) generator, and δ be another $*$-derivation in A with the domain $\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$. Then it follows from Longo [**9**] that δ is *relatively bounded* for δ_α, that is, there are nonnegative constants c and d such that $\|\delta(x)\| \leq c\|x\| + d\|\delta_\alpha(x)\|$ for all x in $\mathscr{D}(\delta_\alpha)$. Furthermore, Batty [**2**] showed that if $d < 1$, $\delta_\beta = \delta_\alpha + \delta$ generates a C^*-dynamics $(A, \beta, \mathbf{R})$, which is said to be *small perturbation* for α. In the special case that $d = 0$, the dynamics β is said to be *bounded perturbation* for α.

Under the above results, we can discuss the stability of physical properties for the initial dynamics under perturbations. For example, it is well known that the existence of KMS-states and ground states is stable under bounded perturbations [**1**, **6** and **11**], and Batty [**3**] showed that the stability of ground states also holds for type-I C^*-dynamical systems and its small perturbations.

In this article we first discuss the covariance of a $*$-derivation δ in A with $\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$ in irreducible α-covariant representations and show that the relative boundedness of δ for δ_α automatically leads that of their implementing selfadjoint operators. As its corollary, we assert that the existence of ground states is stable under sufficiently small perturbations. Our discussions heavily depend on a recent work of Kishimoto [**8**]. The author would like to thank Professor Kishimoto for fruitful discussions with him and helpful suggestions.

We begin by stating the following theorem.

THEOREM 1. *Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system with a generator δ_α, let ψ be an α-invariant pure state of A, and let $(\pi, \mathscr{H}, \xi, H)$ be the G.N.S. α-covariant representation for ψ such that $iH\pi(a)\xi = \pi(\delta_\alpha(a))\xi$. Consider a $*$-derivation δ with the domain*

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$\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$. Then there exists a selfadjoint element k in A such that $\psi \circ (\delta + \mathrm{Ad}\, ik) = 0$ and we can find a selfadjoint operator K in $\mathscr{H}$ satisfying

(i) $\pi(\delta(a)) = [iK, \pi(a)]$ *for all* $a \in \mathscr{D}(\delta_\alpha)$,

(ii) $\mathscr{D}(K) \supset \mathscr{D}(H)$, *and*

(iii) *K is relatively bounded for H, that is,*

$$\|K\eta\| \leq c_1\|\eta\| + c_2\|H\eta\| \quad \textit{for all } \eta \in \mathscr{D}(H),$$

where c_1 and c_2 are nonnegative constants that do not depend on η. Furthermore, if δ is a generator, we have $\mathscr{D}(K) = \mathscr{D}(H)$.

PROOF. We first note that H is defined by $iH\pi(a)\xi = \pi(\delta_\alpha(a))\xi$ for $a \in \mathscr{D}(\delta_\alpha)$ and satisfies $(1 \pm iH)\pi(\mathscr{D}(\delta_\alpha))\xi = \pi((1 \pm \delta_\alpha)\mathscr{D}(\delta_\alpha))\xi = \pi(A)\xi = \mathscr{H}$, so that H is a selfadjoint operator in $\mathscr{H}$ with the domain $\mathscr{D}(H) = \pi(\mathscr{D}(\delta_\alpha))\xi$.

Let $\bar{\alpha}$ be a σ-weakly continuous one-parameter group of *-automorphisms of $M = \mathscr{B}(\mathscr{H})$ defined by $\bar{\alpha}(x) = e^{itH}xe^{-itH}$, then $\Delta_{\bar{\alpha}} = \mathrm{Ad}\, iH$ is its generator. We denote by $M^{\bar{\alpha}}(\Omega)$ (resp. $A^{\alpha}(\Omega)$) a spectral subspace for $\bar{\alpha}$ (resp. α) associated with a set $\Omega \subset \hat{\mathbf{R}} = \mathbf{R}$ and set $M_F^{\bar{\alpha}} = \bigcup M^{\bar{\alpha}}(\Omega)$ (resp. $A_F^{\alpha} = \bigcup A^{\alpha}(\Omega)$), where Ω runs over compact sets in $\mathbf{R}$. Batty [**3**] showed that there exists a *-derivation Δ from $\pi(\mathscr{D}(\delta_\alpha))$ into $\pi(A)$ such that $\Delta(\pi(a)) = \pi(\delta(a))$ for $a \in \mathscr{D}(\delta_\alpha)$. It follows from Kishimoto [**8**] that Δ is σ-weakly closable and its closure $\bar{\Delta}$ is a generator of a σ-weakly continuous dynamics on M whose domain $\mathscr{D}(\bar{\Delta})$ contains $M_F^{\bar{\alpha}}$, and the restriction $\bar{\Delta}|_{M^{\alpha}(\Omega)}$ is σ-weakly continuous on $M^{\bar{\alpha}}(\Omega) \cap M_1$, where M_1 is the unit ball of M and Ω is an arbitrary compact subset of $\mathbf{R}$.

Let p be a one-dimensional projection in M associated with ξ, then p is fixed by the action $\bar{\alpha}$, so that belongs to $\mathscr{D}(\bar{\Delta})$. Set $h = i(\bar{\Delta}(p)p - p\bar{\Delta}(p)) \in M_h$, where M_h denotes the selfadjoint part of M. Since $\pi(A)$ acts irreducibly on $\mathscr{H}$, there exists $k \in A_h$ such that $\pi(k)\xi = h\xi$. Then we have, for x in $\mathscr{D}(\bar{\Delta})$,

$$\begin{aligned}
\big((\bar{\Delta} + \mathrm{Ad}\, i\pi(k))(x)\xi, \xi\big) &= \big(\bar{\Delta}(x)\xi, \xi\big) + ([i\pi(k), x]\xi, \xi)\\
&= \big(\bar{\Delta}(x)\xi, \xi\big) + ([ih, x]\xi, \xi)\\
&= \big((p\bar{\Delta}(x)p - p\bar{\Delta}(p)px + p\bar{\Delta}(p)x + px\bar{\Delta}(p) - xp\bar{\Delta}(p)p)\xi, \xi\big)\\
&= \big(\bar{\Delta}(pxp)\xi, \xi\big)\\
&= \big(p\bar{\Delta}(pxp)p\xi, \xi\big)\\
&= 0,
\end{aligned}$$

where we used the equality $p\bar{\Delta}(p)p = 0$ and $pxp = \lambda p$ for some $\lambda \in \mathbf{C}$, which easily follows from $\bar{\Delta}(p) = \bar{\Delta}(p^2) = p\bar{\Delta}(p) + \bar{\Delta}(p)p$ and the fact that p is one-dimensional. These techniques are due to Bratteli and Robinson [**5**]. Since $\bar{\Delta} + \mathrm{Ad}\, i\pi(k)$ is a generator of a σ-weakly continuous dynamics on M, if we set

$$iK'x\xi = \big(\bar{\Delta} + \mathrm{Ad}\, i\pi(k)\big)(x)\xi,$$

it follows from the above calculation that K' is a selfadjoint operator in $\mathscr{H}$ such that

$\mathscr{D}(K') = \mathscr{D}(\bar{\Delta})\xi$ and $\bar{\Delta} + \mathrm{Ad}\, i\pi(k) = \mathrm{Ad}\, iK'$. Setting $K = K' - \pi(k)$, we have, for $a \in \mathscr{D}(\delta_\alpha)$,

$$\begin{aligned}\pi(\delta(a)) &= \pi((\delta + \mathrm{Ad}\, ik)(a)) - \pi(\mathrm{Ad}\, ik(a))\\ &= \big(\bar{\Delta} + \mathrm{Ad}\, i\pi(k)\big)(\pi(a)) - \mathrm{Ad}\, i\pi(k)(\pi(a))\\ &= \mathrm{Ad}\, iK'(\pi(a)) - \mathrm{Ad}\, i\pi(k)(\pi(a))\\ &= \mathrm{Ad}\, iK(\pi(a)),\end{aligned}$$

so that the assertions (i), (ii), and $\psi \circ (\delta + \mathrm{Ad}\, ik) = 0$ are proved.

By the general theory, $(1 + \Delta_{\bar{\alpha}})^{-1}$ is contractive and σ-weakly continuous and maps the spectral subspace $M^{\bar{\alpha}}(\Omega)$ onto itself. Since $\bar{\Delta}|_{M^{\bar{\alpha}}(\Omega)}$ is σ-weakly continuous on $M^{\bar{\alpha}}(\Omega) \cap M_1$ for every compact subset Ω of $\mathbf{R}$, $\bar{\Delta}(1 + \Delta_{\bar{\alpha}})^{-1}|_{M^{\bar{\alpha}}(\Omega)}$ is also σ-weakly continuous on $M^{\bar{\alpha}}(\Omega) \cap M_1$. On the other hand, by Longo **[9]**, the condition $\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$ leads to the relative boundedness of δ for δ_α, and Batty **[3]** shows that it also holds for Δ and $\Delta_{\bar{\alpha}}$ on $\pi(\mathscr{D}(\delta_\alpha))$, that is, there exist nonnegative constants c and d such that $\|\Delta\pi(a)\| \le c\|\pi(a)\| + d\|\Delta_{\bar{\alpha}}\pi(a)\|$ for all a in $\mathscr{D}(\delta_\alpha)$. Thus we have, for $x \in \pi(A)$,

$$\begin{aligned}\big\|\Delta(1 + \Delta_{\bar{\alpha}})^{-1}x\big\| &\le c\big\|(1 + \Delta_{\bar{\alpha}})^{-1}x\big\| + d\big\|\Delta_{\bar{\alpha}}(1 + \Delta_{\bar{\alpha}})^{-1}x\big\|\\ &\le (c + 2d)\|x\|.\end{aligned}$$

By the continuity of $\bar{\Delta}(1 + \Delta_{\bar{\alpha}})^{-1}$ and the Kaplansky density theorem, the inequality also holds for any x in $M_F^{\bar{\alpha}}$. Setting $y = (1 + \Delta_{\bar{\alpha}})^{-1}x$, we have

$$\begin{aligned}\|\bar{\Delta}y\| &\le (c + 2d)\|(1 + \Delta_{\bar{\alpha}})y\|\\ &\le (c + 2d)\big(\|y\| + \|\Delta_{\bar{\alpha}}(y)\|\big) \quad \text{for all } y \in M_F^{\bar{\alpha}}.\end{aligned}$$

Since $\bar{\Delta}$ is σ-weakly closed, hence norm closed, and A_F^α is the core for δ_α, it follows that

$$\|\bar{\Delta}(\pi(a)p)\| \le (c + 2d)\big(\|\pi(a)p\| + \|\Delta_{\bar{\alpha}}(\pi(a)p)\|\big) \quad \text{for all } a \text{ in } \mathscr{D}(\delta_\alpha).$$

Thus we have, for $a \in \mathscr{D}(\delta_\alpha)$,

$$\begin{aligned}\|K\pi(a)\xi\| &\le \|(K + \pi(k))\pi(a)\xi\| + \|\pi(k)\pi(a)\xi\|\\ &= \|K'\pi(a)\xi\| + \|\pi(k)\pi(a)\xi\|\\ &= \|(\Delta + \mathrm{Ad}\, i\pi(k))(\pi(a))\xi\| + \|\pi(k)\pi(a)\xi\|\\ &= \|(\Delta + \mathrm{Ad}\, i\pi(k))(\pi(a))p\| + \|\pi(k)\pi(a)\xi\|\\ &= \|(\bar{\Delta} + \mathrm{Ad}\, i\pi(k))(\pi(a)p)\| + \|\pi(k)\pi(a)\xi\|\\ &\le (c + 2d)\big(\|\pi(a)p\| + \|\Delta_{\bar{\alpha}}(\pi(a)p)\|\big)\\ &\quad + 2\|k\|\|\pi(a)p\| + \|k\|\|\pi(a)\xi\|\\ &= (c + 2d + 3\|k\|)\|\pi(a)\xi\| + (c + 2d)\|\Delta_{\bar{\alpha}}(\pi(a))p\|\\ &= (c + 2d + 3\|k\|)\|\pi(a)\xi\| + (c + 2d)\|H\pi(a)\xi\|,\end{aligned}$$

where we used $(\bar{\Delta} + \mathrm{Ad}\, i\pi(k))(p) = \Delta_{\bar{\alpha}}(p) = 0$.

If δ is a generator, $\delta + \mathrm{Ad}\, ik$ is also a generator, so that, as the first remark in the proof for $\mathscr{D}(H)$, we have $\mathscr{D}(K') = \mathscr{D}(K) = \pi(\mathscr{D}(\delta_\alpha))\xi$. This completes the proof.

Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system and let δ_α be its generator. A state ψ of A is said to be a *ground state* for α iff it satisfies $i\psi(a^*\delta_\alpha(a)) \leq 0$ for all $a \in \mathscr{D}(\delta_\alpha)$. In this case ψ is automatically α-invariant, if we denote the G.N.S. α-covariant representation for ψ by $(\pi, \mathscr{H}, \xi, H)$, we have $H \geq 0$ [**6**, **11**].

COROLLARY 2. *Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system which has ground states, let δ_α be a generator of α, and let δ be another $*$-derivation in A with the domain $\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$. Then, for every real number λ with $|\lambda|$ sufficiently small, the continuous dynamics β generated by $\delta_\beta = \delta_\alpha + \lambda\delta$ also has ground states.*

PROOF. The set of all ground states for α forms a weak*-closed face of the state space of A, so that its extremal points are pure ground states for α. Let ψ be a pure ground state for α, and let $(\pi, \mathscr{H}, \xi, H)$ be the G.N.S. α-covariant representation for ψ with $H \geq 0$. Then there exists a selfadjoint operator K in $\mathscr{H}$ satisfying the conditions in Theorem 1. It follows that

$$\pi\big(\delta_\beta(a)\big) = \pi\big((\delta_\alpha + \lambda\delta)(a)\big) = \big[i(H + \lambda K), \pi(a)\big] \quad \text{for } a \in \mathscr{D}(\delta_\alpha).$$

For every real number λ with $|\lambda|$ sufficiently small, by [**2**], δ_β generates a continuous dynamics β and, by the relative boundedness of K for H, the Kato-Rellich Theorem [**10**, Theorem 5.4.11] asserts that $H + \lambda K$ is a lowerbounded selfadjoint operator. Thus, by [**3**, **11**], β has ground states, which completes the proof.

COROLLARY 3. *Let $(A, \alpha, \mathbf{R})$ be a C^*-dynamical system which has an irreducible α-covariant representation $(\pi, \mathscr{H}, H)$ with $\pi(\alpha_t(a)) = e^{itH}\pi(a)e^{-itH}$, δ be a $*$-derivation in A with the domain $\mathscr{D}(\delta) = \mathscr{D}(\delta_\alpha)$, where δ_α is the generator of α. Then there exists a selfadjoint operator K in $\mathscr{H}$ satisfying the three conditions* (i)–(iii) *in Theorem* 1. *Moreover, if δ is a generator, we have $\mathscr{D}(K) = \mathscr{D}(H)$.*

PROOF. Set $\bar{\alpha}_t(x) = e^{itH}xe^{-itH}$ and $\Delta_{\bar{\alpha}} = \mathrm{Ad}\, iH$, and consider the C^*-dynamical system $(C(\mathscr{H}), \bar{\alpha}|_{C(\mathscr{H})}, \mathbf{R})$, where $C(\mathscr{H})$ is the C^*-algebra of all compact operators on $\mathscr{H}$. By Bratteli and Robinson [**5**], there exists a rank-one projection p in $\mathscr{D}(\Delta_{\bar{\alpha}})$. Let ξ be a unit vector in the range of p. The irreducibility of π implies that

$$\pi(h)\xi = i(\Delta_{\bar{\alpha}}(p)p - p\Delta_{\bar{\alpha}}(p))\xi \quad \text{for some } h \in A_h.$$

Then, as in the proof of Theorem 1, we can see that

$$\begin{aligned}\psi_\xi((\delta_\alpha + \mathrm{Ad}\, ih)(a)) &= \big(\pi((\delta_\alpha + \mathrm{Ad}\, ih)(a))\xi, \xi\big)\\ &= 0 \quad \text{for } a \in \mathscr{D}(\delta_\alpha).\end{aligned}$$

Applying Theorem 1 to $e^{t(\delta_\alpha + \mathrm{Ad}\, ih)}$ and δ, the assertion follows.

The relative boundedness question has been also considered in a slightly different context in the recent papers [**4** and **7**]. The author would like to thank the referee for having announced these papers to him.

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