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H^1 SUBORDINATION AND EXTREME POINTS

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ABSTRACT. Suppose that F is an element of H^1 (Hardy class of order 1 over the unit disc). Let $s(F)$ denote the set of functions subordinate to F. We show that if ϕ is inner and $\phi(0) = 0$; then $F \circ \phi$ is an extreme point of the closed convex hull of $s(F)$.

1. Introduction. Let $U = \{z: |z| < 1\}$ and let A denote the set of functions analytic in U with the topology given by uniform convergence on compact subsets of U. It is known that A is a metrizable and locally convex space [**8**, p. 1]. Let B denote the subset of A consisting of all functions ϕ that satisfy $|\phi(z)| < 1$ $(z \in U)$.

Suppose that F is a nonconstant function in A. Let $s(F)$ denote the set of functions g that are subordinate to F in U. That is to say, $s(F)$ is the collection of functions g given by

$$g = F \circ \phi$$

where $\phi \in B$ and $\phi(0) = 0$. The closed convex hull of $s(F)$ is denoted by $\mathrm{Hs}(F)$ and the set of extreme points of $\mathrm{Hs}(F)$ is denoted by $\mathrm{Ex}(F)$. $\mathrm{Ex}(F) \subseteq s(F)$ because $s(f)$ is compact [**2**, p. 440].

A function $f \in A$ is said to belong to the class H^p $(0 < p < \infty)$ if

$$\|f\|_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

Each $f \in H^p$ has a radial limit $f(e^{i\theta})$ almost everywhere and $f \in L^p$. For $f \in H^p$, we also have

$$\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta \underset{r \to 1}{\to} 0$$

and

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leqslant \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta$$

for every $0 < r < 1$ [**3**, p. 21].

It is known that if $F \in H^p$ and $f \in \mathrm{Hs}(F)$ then $\|f\|_p < \|F\|_p$ [**5**, p. 465].

Suppose that $F \in H^p$ and $f \in s(F)$. In [**7**, p. 351] Ryff showed that

$$\|f\|_p = \|F\|_p \quad \text{if and only if} \quad f = F \circ \phi$$

where ϕ is inner ($\phi \in B$ and $|\phi(e^{i\theta})| = 1$ almost everywhere) and $\phi(0) = 0$.

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In [**5**, p. 465] it was proven that if $F \in H^p$ $(p > 1)$ then $\{F \circ \phi\colon \phi$ inner, $\phi(0) = 0\} \subset \operatorname{Ex}(F)$. D. J. Hallenbeck (unpublished) extended this result to the case $p = 1$ when either F is outer or F is univalent.

In this paper, we prove the following

THEOREM. *If* $F \in H^1$ *then*

$$\{F \circ \phi\colon \phi \text{ inner}, \phi(0) = 0\} \subset \operatorname{Ex}(F). \tag{1}$$

2. Lemma.

LEMMA. *Suppose that* X *is a nonempty subset of* A *and there is a number* $M > 0$ *so that* $\|g\|_1 \leqslant M$ *for every* $g \in X$. *Let* μ *be a probability measure on* X *and* L_r *be the function on* $X \times [0, 2\pi]$ *defined by*

$$L_r(g, \theta) = g(re^{i\theta}) \qquad (r \leqslant 1).$$

Then L_r *is measurable and integrable on* $X \times [0, 2\pi]$ *with respect to* $d\mu\, d\theta$.

REMARK. μ is a Borel measure with $\mu(X) = 1$.

PROOF. First we want to show that L_r is continuous when $r < 1$. Let (g_n, θ_n) be a sequence in $X \times [0, 2\pi]$ which converges in the product topology to $(g, \theta) \in X \times [0, 2\pi]$. Then $g_n \to g$, $g_n' \to g'$ uniformly on compact subsets of U and $e^{i\theta_n} \to e^{i\theta}$. This implies that for every $\varepsilon > 0$ there exists an integer $N > 0$ so that whenever $n > N$

$$|g(re^{i\theta}) - g_n(re^{i\theta})| < \varepsilon, \qquad |g_n(re^{i\theta}) - g_n(re^{i\theta_n})| < (\varepsilon + k)\varepsilon$$

where $k = \max_{|z| \leqslant r} |g'(z)|$. Since

$$|g(re^{i\theta}) - g_n(re^{i\theta_n})| \leqslant |g(re^{i\theta}) - g_n(re^{i\theta})| + |g_n(re^{i\theta}) - g_n(re^{i\theta_n})|$$

we conclude that $L_r(g, \theta)$ is continuous.

Second, we want to show that L_r is measurable. For $r < 1$ and α real, let

$$E_\alpha = \{(g, \theta)\colon \operatorname{Re} L_r(g, \theta) > \alpha\}.$$

E_α is open because $\operatorname{Re} L_r$ $(r < 1)$ is continuous. Since the spaces X and $[0, 2\pi]$ are separable (polynomials whose coefficients have rational real parts and rational imaginary parts are dense in A), $X \times [0, 2\pi]$ is separable and every open set can be written as a countable union of sets of the form $O_n \times I_n$, where O_n is open in X and I_n is open in $[0, 2\pi]$. Hence E_α is measurable and consequently $\operatorname{Re} L_r$ is measurable. Similarly, it can be shown that $\operatorname{Im} L_r$ is also measurable. Hence L_r $(r < 1)$ is measurable and consequently, as $L_1 = \lim_{r \to 1} L_r$, L_1 is also measurable.

Now, we want to show that L_r is integrable. We have, by Tonelli's theorem, that

$$\int\!\!\int |g(re^{i\theta})|\, d\mu\, d\theta = \int\!\!\int |g(re^{i\theta})|\, d\theta\, d\mu \leqslant 2\pi M \qquad (r \leqslant 1).$$

Hence L_r is integrable.

3. Proof of the theorem. Write $F = I \cdot G$, where I is inner and G is outer [**4**, p. 74], and assume, without loss of generality, that $\|F\|_1 = 1$. Let $f = F \circ \phi$, where ϕ is inner and $\phi(0) = 0$. It is known that $G(\phi)$ is outer [**9**, p. 260] and $I(\phi)$ is inner. Since

$\mathrm{Hs}(F)$ is a metrizable and compact convex subset of A, it follows that $\mathrm{Ex}(F)$ is a G_δ subset of $\mathrm{Hs}(F)$ [**6**, p. 7] and, in addition, by Choquet's theorem [**6**, p. 19], there is a probability measure μ on $\mathrm{Hs}(F)$ supported by $\mathrm{Ex}(f)$ so that

$$f = \int_{\mathrm{Ex}(F)} g \, d\mu(g)$$

and, for every continuous linear functional L on A,

$$L(f) = \int_{\mathrm{Ex}(F)} L(g) \, d\mu(g).$$

This implies that $f(z) = \int_{\mathrm{Ex}(F)} g(z) \, d\mu(g)$ $(z \in U)$. Hence

$$|f(z)| \leqslant \int_{\mathrm{Ex}(F)} |g(z)| d\mu(g) \qquad (z \in U)$$

and, by Lemma 2, as $\|g\|_1 \leqslant 1$ for every $g \in \mathrm{Hs}(F)$,

$$\begin{aligned} \frac{1}{2\pi} \int |f(re^{i\theta})| d\theta &\leqslant \frac{1}{2\pi} \int_{\mathrm{Ex}(F)} \int_0^{2\pi} |g(re^{i\theta})| d\theta \, d\mu(g) \\ &\leqslant \int_{\mathrm{Ex}(F)} \|g\|_1 \, d\mu(g) \leqslant 1. \end{aligned}$$

Let $r \to 1$, to conclude that

$$1 = \|f\|_1 \leqslant \int_{\mathrm{Ex}(F)} \|g\|_1 \, d\mu(g) \leqslant 1.$$

Therefore $\|g\|_1 = 1$, μ-almost everywhere. Since $\mathrm{Ex}(F) \subset \mathrm{s}(F)$, it follows by Ryff's theorem, that, for μ-almost every $g \in \mathrm{Ex}(F)$, $g = F \circ \psi$ where ψ is inner and $\psi(0) = 0$. Consequently, $G \circ \psi$ is outer [**9**, p. 260] and $I \circ \psi$ is inner.

Now, we claim that $f(e^{i\theta}) = \int_{\mathrm{Ex}(F)} g(e^{i\theta}) \, d\mu(g)$ for almost all θ. To show this, we let

$$H_r(g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta}) - g(e^{i\theta})| d\theta.$$

Then $\lim_{r \to 1} H_r(g) = 0$, $H_r(g) \leqslant 2\|g\|_1 \leqslant 2$ and, by the lemma and Tonelli's theorem, H_r is integrable. This and the bounded convergence theorem give $\lim_{r \to 1} \int_{\mathrm{Ex}(F)} H_r(g) \, d\mu(g) = 0$. Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) - \int_{\mathrm{Ex}(F)} g(e^{i\theta}) \, d\mu \right| d\theta \leqslant \int_{\mathrm{Ex}(F)} H_r(g) \, d\mu(g) \underset{r \to 1}{\to} 0.$$

Since $\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})| \, d\theta \underset{r \to 1}{\to} 0$, $f(e^{i\theta}) = \int_{\mathrm{Ex}(F)} g(e^{i\theta}) \, d\mu(g)$ for almost all θ and the claim is proved.

Let L be the linear functional on H^1 defined by

$$L(h) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta})|}{f(e^{i\theta})} h(e^{i\theta}) \, d\theta \tag{2}$$

where $h \in H^1$. Then $|L(h)| \leqslant 1$ whenever $\|h\|_1 \leqslant 1$, in particular $|L(h)| \leqslant 1$ for $h \in \mathrm{Hs}(F)$, and

$$1 = L(f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta})|}{f(e^{i\theta})} \int_{\mathrm{Ex}(F)} g(e^{i\theta})\, d\mu(g)\, d\theta. \tag{3}$$

Since $(|f(e^{i\theta})|/f(e^{i\theta}))g(e^{i\theta})$ is measurable on $\mathrm{Hs}(F) \times [0, 2\pi]$ and $|g(e^{i\theta})|$ is integrable on $\mathrm{Hs}(F) \times [0, 2\pi]$, by the lemma, it follows that $(|f(e^{i\theta})|/f(e^{i\theta}))g(e^{i\theta})$ is integrable on $\mathrm{Hs}(F) \times [0, 2\pi]$. Hence

$$1 = L(f) = \int_{\mathrm{Ex}(F)} L(g)\, d\mu(g)$$

and consequently $L(g) = 1$, μ-almost everywhere, because $|L(g)| \leqslant 1$. This and (2) give that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta})|}{f(e^{i\theta})} g(e^{i\theta})\, d\theta = \|g\|_1 = 1, \quad \mu\text{-almost everywhere.}$$

Hence, it follows that $g(e^{i\theta})/f(e^{i\theta}) = K_g(e^{i\theta}) > 0$ for almost every θ and μ-almost every $g \in \mathrm{Ex}(F)$. We also have, as $f(e^{i\theta}) = \int_{\mathrm{Ex}(F)} g(e^{i\theta})\, d\mu(g)$,

$$\int_{\mathrm{Ex}(F)} K_g(e^{i\theta})\, d\mu(g) = 1 \quad \text{for almost all } \theta.$$

Since $G(\phi)$ is the outer factor of f and $G(\psi)$ is the outer factor of g, for μ-almost every $g \in \mathrm{Ex}(F)$, where $g = F \circ \psi$ with ψ inner and $\psi(0) = 0$, it follows that

$$\int_0^{2\pi} \log|f(e^{i\theta})| d\theta = \log|G(\phi(0))| = \log|G(0)|$$

and

$$\int_0^{2\pi} \log|g(e^{i\theta})| d\theta = \log|G(\psi(0))| = \log|G(0)| \quad [\mathbf{3}, \text{p. } 24].$$

Therefore $\int_0^{2\pi} \log K_g(e^{i\theta})\, d\theta = 0$ for μ-almost all $g \in \mathrm{Ex}(F)$. But then by Jensen's inequality,

$$1 = \exp\int_0^{2\pi} \log K_g(e^{i\theta}) \frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} K_g(e^{i\theta})\, \frac{d\theta}{2\pi} \tag{4}$$

for μ-almost all $g \in \mathrm{Ex}(F)$. Hence

$$1 = \int_{\mathrm{Ex}(F)} \left[\exp\int_0^{2\pi} \log K_g(e^{i\theta})\, \frac{d\theta}{2\pi}\right] d\mu(g)$$

$$\leqslant \int_{\mathrm{Ex}(F)} \int_0^{2\pi} K_g(e^{i\theta})\, \frac{d\theta}{2\pi} d\mu = \int_0^{2\pi} \int_{\mathrm{Ex}(F)} K_g(e^{i\theta})\, d\mu\, \frac{d\theta}{2\pi} = 1.$$

This and (4) imply that $(1/2\pi)\int_0^{2\pi} K_g(e^{i\theta})\, d\theta = 1$ for μ-almost all $g \in \mathrm{Ex}(F)$. Since exp is strictly convex, we conclude from (4) that $K_g(e^{i\theta}) = 1$ for almost all θ and μ-almost all $g \in \mathrm{Ex}(F)$. Therefore f is an extreme point and μ is a point mass.

REMARKS. (1) The above proof is partially a generalization of a proof due to K. de Leeuw and W. Rudin [**4**, p. 158].

(2) For any $p < 1$, choose λ so that $p < \lambda < 1$. Then $F(z) = 1/(1 - z)^{\lambda/p} \in H^p$ and since $\lambda/p > 1$ it is known that $\mathrm{Ex}(F) = \{F(yz): |y| = 1\}$ [**1**]. It follows that once $p < 1$ the inclusion in the above theorem is false (see (1)). This remark is due to D. J. Hallenbeck.

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