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THE FORELLI PROBLEM CONCERNING IDEALS IN THE DISK ALGEBRA $A(\mathbf{D})$

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ABSTRACT. Let $Z(f)$ be the zero set of a function $f \in A(\mathbf{D})$ and $Z(I) = \bigcap_{f \in I} Z(f)$ the zero set of an ideal I in $A(\mathbf{D})$. It is shown that in the disk algebra $A(\mathbf{D})$ every finitely generated ideal I has the weak Forelli property, i.e. there exists a function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$, where T is the boundary of the unit circle $\mathbf{D}$. On the other hand, there exists a finitely generated ideal I in $A(\mathbf{D})$ such that $Z(f) \neq Z(I)$ for each choice of $f \in I$. This provides us with a negative answer to a problem of F. Forelli [1].

1. The disk algebra $A(\mathbf{D})$ is the Banach algebra of all those continuous functions in the closed unit disk $\overline{\mathbf{D}}$ which are analytic in the open unit disk $\mathbf{D}$, under the usual pointwise algebraic operations and the supremum norm.

Let $Z(f) = \{z \in \overline{\mathbf{D}}: f(z) = 0\}$ denote the zero set of a function $f \in A(\mathbf{D})$ and $Z(I) = \bigcap_{f \in I} Z(f)$ the zero set of an ideal I in $A(\mathbf{D})$.

In one of his papers F. Forelli [1] posed the problem of classifying those ideals in $A(\mathbf{D})$ which have the property that there exists a function $f \in I$ such that the zero set of f agrees with the zero set of the ideal I, i.e. for which ideals I do we have $Z(f) = Z(I)$ for a function $f \in I$? Such a property will be referred to as the "Forelli property."

It is known that every closed ideal in $A(\mathbf{D})$ has this property. On the other hand, it remained hitherto unsolved whether each finitely generated ideal in $A(\mathbf{D})$ has the Forelli property (Forelli [1, p. 389]). We now solve this problem. The answer, however, will be negative; a fact that was not expected in view of the results in the ring $H(\mathbf{D})$ of all analytic functions in the unit disk $\mathbf{D}$.

DEFINITION 1. An ideal I in the disk algebra $A(\mathbf{D})$ has the "Forelli property" if there exists a function $f \in I$ such that $Z(f) = Z(I)$.

PROPOSITION 1. *There exists a finitely generated ideal in the disk algebra which does not have the Forelli property.*

PROOF. As generators we take the functions $f_i(z) = (1 - z)B_i(z)$ $(i = 1, 2)$ of v. Renteln [5, p. 139], where $a_n = 1 - n^{-2}$ are the zeros of the Blaschke product B_1 and $b_n = a_n + \varepsilon_n$, $\varepsilon_n = n^{-2}\exp[-(1 - a_n)^{-2}] = n^{-2}\exp(-n^4)$, the zeros of the Blaschke product B_2. Let $I = (f_1, f_2)$ denote the corresponding ideal. It is now

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obvious that $Z(I) = \{1\}$. Hence it is sufficient to show that each function $f \in I$ has infinitely many zeros in $\mathbf{D}$.

Let $f = g_1 f_1 + g_2 f_2$ belong to the ideal I. By the factorization theorem of F. Riesz, we have $f = Bg$, where B is a Blaschke product and g is a function in $A(\mathbf{D})$ which vanishes nowhere in $\mathbf{D}$. Then $B = h_1 f_1 + h_2 f_2$, where h_1 and h_2 are functions of the Nevanlinna class N.

This yields the following estimate (see [**5**]):

$$\begin{aligned} |B(a_n)| &= |h_2(a_n)|\,|f_2(a_n)| \\ &\leqslant 2\exp\left(\frac{C}{1-|a_n|}\right)\frac{\varepsilon_n}{1-|a_n|} = 2\exp(Cn^2)\exp(-n^4) \end{aligned}$$

where $C > 0$ is a constant that is independent of n.

Thus $\lim_{n\to\infty}|B(a_n)| = 0$. This proves that B has infinitely many zeros in $\mathbf{D}$, i.e. every member of I vanishes infinitely often in $\mathbf{D}$. □

2. Before we proceed, we have to give an auxiliary result of Davie, Gamelin and Garnett.

Let E be an open subset of T, the boundary of the unit circle $\mathbf{D}$. We denote by L_E^∞ the set of all functions in L^∞ that are continuous in E and by H_E^∞ the set of all bounded analytic functions in $\mathbf{D}$ that are continuously extendable to E.

LEMMA (SEE GARNETT [**2**, P. 399, EX. 15c]). *Every function of L_E^∞ can be uniformly approximated by a quotient of the form*

$$\frac{\sum_{i=1}^N c_i B_i}{B},$$

where B_i $(i = 1,\dots,N)$ and B are Blaschke products in H_E^∞ and $c_i \in \mathbf{C}$.

Our preceding results in §1 show that one cannot expect that even a finitely generated ideal I in the disk algebra has the Forelli property, i.e. that I contains a function f such that $Z(f) = Z(I)$. Therefore, we are going to modify the question of Forelli by considering merely the boundary zero sets. This leads us to the following definition.

DEFINITION 2. An ideal I in the disk algebra $A(\mathbf{D})$ has the "weak Forelli property" if there exists a function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$.

We are now able to prove the main result of this paper.

THEOREM. *Every finitely generated ideal in the disk algebra has the weak Forelli property.*

PROOF. Let $I = (f_1,\dots,f_N)$ be a finitely generated ideal in $A(\mathbf{D})$. Our goal is the construction of a function

$$f = \sum_{i=1}^N H_i f_i \in I$$

such that $|\sum_{i=1}^N H_i f_i| \geqslant C\sum_{i=1}^N |f_i|^2$ holds on T, where $C > 0$ is a constant. This immediately yields $Z(f) \cap T \subset Z(I) \cap T$, and hence the assertion, because the inclusion $Z(f) \cap T \supset Z(I) \cap T$ holds trivially for each function $f \in I$.

Let $g_i = f_i^2$, $E = T \setminus Z(I)$. The inequality of Cauchy and Schwarz yields the estimate

$$(1) \qquad \left| \sum_{i=1}^{N} \frac{\overline{g_i}}{\sum_{n=1}^{N} |g_n|} g_i \right| \geqslant \frac{1}{N} \sum_{i=1}^{N} |g_i| \quad \text{on } E.$$

Because the sum $\sum_{i=1}^{N} |g_i|$ does not vanish in E, the functions $q_i = \overline{g_i}/(\sum_{i=1}^{N} |g_n|)$ are in L_E^∞ $(i = 1, \ldots, N)$.

We are now going to approximate the functions q_i as in the Lemma. More precisely, there exist functions $h_i \in H_E^\infty$ and Blaschke products $B_i \in H_E^\infty$ such that

$$(2) \qquad \|q_i - h_i/B_i\|_\infty \leqslant 1/2N,$$

where $\|\cdot\|_\infty$ is the supremum norm in L^∞. Thus we have by (1) and (2) the following estimates on E:

$$(3) \qquad \left| \sum_{i=1}^{N} \frac{h_i}{B_i} g_i \right| \geqslant \left| \sum_{i=1}^{N} q_i g_i \right| - \sum_{i=1}^{N} \left| q_i - \frac{h_i}{B_i} \right| |g_i|$$
$$\geqslant \left(\frac{1}{N} - \frac{1}{2N} \right) \sum_{i=1}^{N} |g_i|.$$

To get rid of the denominators on the left side of inequality (3), we multiply with $|B_1| \cdot \cdots \cdot |B_N| = 1$. Thus,

$$(4) \qquad \left| \sum_{i=1}^{N} (\varphi_i h_i) g_i \right| \geqslant \frac{1}{2N} \sum_{i=1}^{N} |g_i| \quad \text{on } E,$$

where $\varphi_i = \prod_{j=1; j \neq i}^{N} B_j$ $(i = 1, \ldots, N)$.

By construction, the functions $\varphi_i h_i f_i$ belong to H_E^∞. Because the f_i are continuous on T and vanish in $Z(I)$, the functions $H_i = \varphi_i h_i f_i \in A(\mathbf{D})$ $(i = 1, \ldots, N)$. Thus, $\sum_{i=1}^{N} H_i f_i \in I$ and (4) implies

$$\left| \sum_{i=1}^{N} H_i f_i \right| \geqslant \frac{1}{2N} \sum_{i=1}^{N} |f_i|^2 \quad \text{on } T(!)$$

which was to be proved. □

REMARK. Our Theorem does not hold for every given ideal in the disk algebra. There even exist countably generated ideals which do not have the weak Forelli-property, as the following example shows.

EXAMPLE. Let $\{z_n\}$ be a sequence of different points in T such that $z_n \to 1$ $(n \to \infty)$ and $Z_N = \{z_n : n \geqslant N\} \cup \{1\}$. Because the Z_N are closed sets of measure zero, there exist by a theorem of Fatou (Hoffman [**4**, p. 80]) functions $f_N \in A(\mathbf{D})$ such that $Z(f_N) = Z_N$.

Let $I = (f_1, f_2, \ldots)$ be the ideal generated by the functions $f_1, f_2, \ldots$ in $A(\mathbf{D})$. It is now obvious that $Z(I) \cap T = \{1\}$. But, on the other hand, each function of the ideal I has the fom $f = \sum_{i=1}^{N} g_i f_i$ for an integer N and suitable chosen functions $g_i \in A(\mathbf{D})$. This implies that $Z_N \subset Z(f)$. Thus every function $f \in I$ has infinitely many zeros on T. Consequently, there exists no function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$. □

With regard to the Forelli problem, we are also going to investigate the prime ideals in the disk algebra. Whereas the maximal ideals in $A(\mathbf{D})$ satisfy the Forelli property, there exist, on the other hand, prime ideals which do not have even the weak Forelli property.

PROPOSITION 2. *In the disk algebra there exist prime ideals which do not have the weak Forelli property.*

PROOF. First we define the set $M = \{f \in A(\mathbf{D})\colon z = 1$ is not a cluster point of the zeros of the function f on $T\}$.

It is obvious that M is closed under multiplication. Let $I = (f_1, f_2, \ldots)$ be the ideal of the preceding example. By construction, the ideal I does not intersect M. By the Lemma of Zorn there exists an ideal $P \supset I$ that is maximal relative to $P \cap M = \varnothing$. This ideal is now prime (see [3, p. 6]). Moreover, the zero set $Z(P) = \{1\}$.

But for every function $f \in P$, the point $z = 1$ is a cluster point of the zeros of f on T, because $P \cap M = \varnothing$. Thus there exists no function in P such that $Z(f) \cap T = Z(P) \cap T$. $\square$

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REFERENCES

1. F. Forelli, *A note on the ideals in the disk algebra*, Proc. Amer. Math. Soc. **84** (1982), 389–392. MR **83c** 46041.

2. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981. MR **83g** 30037

3. L. Gillman and M. Jerison, *Rings of continuous functions*, Graduate Texts in Math., No. 43, Springer-Verlag, Berlin and New York, 1976. MR **53** #11352.

4. K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, N. J., 1962. MR **24** #A 2844.

5. M. v. Renteln, *Every subring R of N with $A(\mathbf{D}) \subset N$ is not adequate*, Acta Sci. Math. **39** (1977), 139–140. MR **56** #5908.

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