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WEIGHTED REVERSE WEAK TYPE INEQUALITIES FOR THE ERGODIC MAXIMAL FUNCTION AND THE CLASSES $L\log^+ L$

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ABSTRACT. D. Ornstein proved that the ergodic maximal function satisfies a reverse weak type inequality, and from this he deduced that the integrability of the maximal function f^* implies that f belongs to $L\log^+ L$. Weighted analogues of these results are proved.

Let (Ω, Σ, μ) denote a probability space and let $T\colon \Omega \to \Omega$ be an invertible, measure preserving ergodic transformation. The Maximal Ergodic Theorem asserts that the maximal function f^* defined for nonnegative $f \in L^1(\mu)$ by

$$f^*(x) = \sup_{m,n\geqslant 0} \frac{1}{m+n+1} \sum_{k=-m}^{n} f(T^k x), \qquad x \in \Omega,$$

satisfies the weak type inequality

$$\mu\{x : f^*(x) > \lambda\} \leqslant \lambda^{-1} \int_{\{x : f^*(x) > \lambda\}} f(x)\, d\mu(x) \leqslant \lambda^{-1} \int_\Omega f(x)\, d\mu(x)$$

for all $\lambda > 0$. On the other hand, D. Ornstein [**6**] has shown that f^* also satisfies the reverse weak type inequality

$$\mu\{x : f^*(x) > \lambda\} \geqslant (2\lambda)^{-1} \int_{\{x : f^*(x) > \lambda\}} f(x)\, d\mu(x)$$

for all λ such that $\mu\{x : f^*(x) > \lambda\} < 1$; in particular, this is the case if $\lambda > \lambda_f = \int_\Omega f(x)\, d\mu(x)$. From this he deduced that if f^* is integrable then f belongs to the class $L\log^+ L$. For elementary proofs of these results see the recent papers of R. L. Jones [**3** and **4**]. The purpose of this paper is to prove the following weighted version of Orstein's result.

THEOREM. *Suppose u and v are nonnegative measurable functions on Ω with $u \in L^1(\mu)$. The following statements are equivalent*:

(i) *There is a constant C independent of f such that*

$$(1) \qquad \int_{\{x : f^*(x) > \lambda\}} u(x)\, d\mu(x) \geqslant (C\lambda)^{-1} \int_{\{x : f^*(x) > \lambda\}} f(x) v(x)\, d\mu(x)$$

holds for all λ such that $\mu\{x : f^*(x) > \lambda\} < 1$.

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(ii) *There is a constant D such that*

$$v(x) \leqslant D\frac{1}{2n+1}\sum_{k=-n}^{n} u(T^k x) \tag{2}$$

holds for all $n \geqslant 0$ *and almost all* $x \in \Omega$.

COROLLARY. *Suppose u and v satisfy* (2) *and that* $\int_\Omega f^*(x)u(x)\,d\mu(x) < \infty$. *Then f satisfies*

$$\int_\Omega [f(x)\log^+ f(x)]\,v(x)\,d\mu(x) < \infty.$$

An analogue of the Theorem for the Hardy-Littlewood maximal function in R^n was recently obtained by B. Muckenhoupt [**5**].

Examples of weight function pairs which satisfy (2) may be constructed as follows. A nonnegative weight function ω is said to satisfy the A_1 condition [**2, 7**] if there is a constant C such that $\omega^*(x) \leqslant C\omega(x)$ a.e. in Ω. If $g \in L^1$ and $0 < \delta < 1$, it is shown in [**7**] that $\omega(x) = g^*(x)^\delta$ belongs to A_1. Now if ω belongs to A_1, the Schwarz inequality for sums shows that

$$\begin{aligned}(2n+1)^2 &\leqslant \left\{\sum_{k=-n}^{n}\omega(T^k x)\right\}\left\{\sum_{k=-n}^{n} 1/\omega(T^k x)\right\}\\ &\leqslant (2n+1)\,\omega^*(x)\left\{\sum_{k=-n}^{n} 1/\omega(T^k x)\right\}\\ &\leqslant C(2n+1)\,\omega(x)\left\{\sum_{k=-n}^{n} 1/\omega(T^k x)\right\}.\end{aligned}$$

Thus (2) is satisfied if $u(x) = v(x) = 1/\omega(x)$.

Let Z denote the set of integers and, for any finite subset I of Z, let $|I|$ denote its cardinality. If I consists of finitely many consecutive integers we say that I is an interval. If I is an interval, we write $2I$ for the largest interval containing I in its center and satisfying $|2I| \leqslant 2|I|$. Thus $2I$ is obtained by adjoining to I two intervals (possibly empty) of equal cardinality, one of which immediately precedes and one of which immediately succeeds I. As usual, if E is any set, $\chi_E(x)$ denotes the characteristic function of E.

We need the following lemma.

LEMMA. *Let I be an interval in Z and let J denote the complement of I in Z. Then I is the union of N pairwise disjoint intervals* I_j *which satisfy*

(i) $|I_j|/2 \leqslant \operatorname{dist}(I_j, J) \leqslant 4|I_j|$, $1 \leqslant j \leqslant N$,

(ii) $\sum_j \chi_{2I_j}(n) \leqslant 4\chi_I(n)$ *for all* $n \in Z$.

PROOF OF THE LEMMA. Suppose $|I| > 2$. Otherwise the Lemma is trivially satisfied by choosing intervals I_j, each consisting of one integer. Let I_1 be the largest interval centered in I with $|I_1| \leqslant |I|/2$. Then $I \setminus I_1$ consists of two intervals, say I' and I'' with $|I'| = |I''| \geqslant |I|/4$. We shall show that I' is a union of pairwise disjoint intervals I_j, $j = 2, \ldots, N'$, such that (i) holds. If $|I'| \leqslant 2$ we may write I' as a union of intervals each consisting of one integer and our selection procedure terminates; otherwise I' is the union of two intervals I'_- and I'_+ with $d(I'_-, J) = 1$ and $|I'_+| - 1 \leqslant |I'_-| \leqslant |I'_+|$. Then I'_+ is the union of two intervals I_2 and I_3 satisfying $|I_2| - 1 \leqslant |I_3| \leqslant |I_2|$ and $\operatorname{dist}(I_3, I'_-) = 1$. Clearly, (i) is satisfied for $j = 2, 3$. This selection procedure is now repeated with I'_- in place of I' and, continuing in this way, I' is eventually exhausted by the intervals $I_j, j = 2, \ldots, N'$. It is easy to see that any element of I belongs to at most 3 of the intervals $2I_j$, $2 \leqslant j \leqslant N'$, and that all such $2I_j$ are disjoint from I''. A similar construction is applied to express I'' as a union of intervals $I_j, j = N' + 1, \ldots, N$. The intervals $I_j, j = 1, \ldots, N$, so constructed clearly satisfy (ii).

PROOF OF THE THEOREM. Suppose that (1) holds. We will show that (2) holds with $D = 2C$. Let $A \in \Sigma$ with $0 < \mu(A) < 1/2$. Then $(\chi_A)^*(x) < 1$ if $x \notin A$, so the theorem of dominated convergence together with (1) shows

$$\begin{aligned}\int_A u(x)\,d\mu(x) &= \lim_{\lambda \to 1-} \int_{\{x:\,(\chi_A)^*(x) > \lambda\}} u(x)\,d\mu(x)\\ &\geqslant \lim_{\lambda \to 1-} \frac{1}{C\lambda} \int_{\{x:\,(\chi_A)^*(x) > \lambda\}} \chi_A(x) v(x)\,d\mu(x)\\ &= \frac{1}{C} \int_A v(x)\,d\mu(x).\end{aligned}$$

Since A is arbitrary, this shows that $u(x) \geqslant v(x)/C$ a.e. Thus (2) holds with $D = 2C$ for the case $n = 0$. Now let $n \geqslant 1$. If $A \in \Sigma$ with $\mu(A) > 0$ there is a subset B of A with $\mu(B) > 0$ such that $T^j(B)$, $j = -n, \ldots, n$, are pairwise disjoint. If $x \notin \bigcup_{j=-n}^{n} T^j(B)$, then any 1 which occurs in the sequence $\chi_B(T^k x)$ is both followed and preceded by at least n consecutive zeros; furthermore, $\chi_B(T^k x) = 0$ for $-n \leqslant k \leqslant n$. For such x it then follows that $(\chi_B)^*(x) \leqslant 1/(n+1)$ and hence $\{x: (\chi_B)^*(x) > 1/(n+1)\}$ is a subset of $\bigcup_{j=-n}^{n} T^j(B)$. Thus, (1) implies

$$\begin{aligned}\int_B \sum_{j=-n}^{n} u(T^j x)\,d\mu(x) &= \int_{\bigcup_{j=-n}^{n} T^j(B)} u(x)\,d\mu(x)\\ &\geqslant \int_{\{x:\,(\chi_B)^*(x) > 1/(n+1)\}} u(x)\,d\mu(x)\\ &\geqslant \frac{n+1}{C} \int_{\{x:\,(\chi_B)^*(x) > 1/(n+1)\}} \chi_B(x) v(x)\,d\mu(x)\\ &= \frac{n+1}{C} \int_B v(x)\,d\mu(x).\end{aligned}$$

Since A was arbitrary, it follows that (2) holds with $D = 2C$ for $n \geqslant 1$.

Suppose now that (2) holds. We will show that (1) holds with $C = 20D$.

Observe first that if I is any interval, then (2) implies that

$$\sum_{j \in 2I} u(T^j x) \geqslant D^{-1} |I| v(T^k x) \quad \text{a.e.} \tag{3}$$

for all $k \in I$.

Let $\lambda > \lambda_f$ and $E = \{x: f^*(x) > \lambda\}$. Since T is ergodic it follows that for almost all $x \in E$ there are positive integers $r = r(x)$ and $s = s(x)$ such that $T^j x \in E$ if $-r + 1 \leqslant j \leqslant s - 1$ but $T^j x \notin E$ for $j = -r$ and $j = s$. For each positive integer i let $R_i = \bigcup_{j=0}^{i-1} T^j(B_i)$ where $B_i = \{x \in E: r(x) = 1 \text{ and } s(x) = i\}$. Then $\{R_i\}$ is a sequence of pairwise disjoint subsets of E, and for almost all $x \in E$ there is i, namely $i = r(x) + s(x) - 1$, such that $x \in R_i$. Thus, $E = \bigcup_{i=1}^{\infty} R_i$ and

$$\text{(4)} \qquad \int_E f(x)v(x)\,d\mu(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \int_{B_i} f(T^j x) v(T^j x)\,d\mu(x).$$

Let i be fixed and let I_j denote the intervals generated by the Lemma for the interval $I = \{0, \ldots, i-1\}$. Then (3) shows that

$$\text{(5)} \qquad \begin{aligned} \sum_{j=0}^{i-1} f(T^j x) v(T^j x) &= \sum_j \sum_{k \in I_j} f(T^k x) v(T^k x) \\ &\leqslant \sum_j \sum_{k \in I_j} f(T^k x) \left(D|I_j|^{-1} \sum_{m \in 2I_j} u(T^m x) \right). \end{aligned}$$

Now part (i) of the Lemma shows that for each j there is an interval K_j with $|K_j| \leqslant 5|I_j|$ containing I_j and either -1 or i. Since neither $T^{-1}x$ nor $T^i x$ belongs to E, it follows that

$$\sum_{k \in I_j} f(T^k x) \leqslant \sum_{k \in K_j} f(T^k x) \leqslant |K_j|\lambda \leqslant 5|I_j|\lambda.$$

Using this on the right side of (5) shows, together with (ii) of the Lemma, that

$$\sum_{j=0}^{i-1} f(T^j x) v(T^j x) \leqslant 5D\lambda \sum_j \sum_{m \in 2I_j} u(T^m x) \leqslant 20D\lambda \sum_{m=0}^{i-1} u(T^m x)$$

and, hence,

$$\begin{aligned} \int_{R_i} f(x)v(x)\,d\mu(x) &= \sum_{j=0}^{i-1} \int_{B_i} f(T^j x) v(T^j x)\,d\mu(x) \\ &\leqslant 20D\lambda \sum_{m=0}^{i-1} \int_{B_i} u(T^m x)\,d\mu(x) = 20D\lambda \int_{R_i} u(x)\,d\mu(x). \end{aligned}$$

In view of (4), summing this over the index i yields (1) with $C = 20D$.

PROOF OF THE COROLLARY. Observe first that (2) implies $v(x) \leqslant Du(x)$ a.e. and thus we have

$$\text{(6)} \qquad \begin{aligned} \int_\Omega f(x)v(x)\,d\mu(x) &\leqslant \int_\Omega f^*(x)v(x)\,d\mu(x) \\ &\leqslant D\int_\Omega f^*(x)u(x)\,d\mu(x) < \infty. \end{aligned}$$

Now if $\lambda > \lambda_f$ then $\mu\{x: f^*(x) > \lambda\} < 1$, so (1) shows that

$$\int_\Omega \left[f(x) \log^+\left(\frac{f(x)}{\lambda_f} \right) \right] v(x)\, d\mu(x) = \int_{\lambda_f}^\infty \frac{d\lambda}{\lambda} \int_{\{x: f(x)>\lambda\}} f(x) v(x)\, d\mu(x)$$

is bounded above by

$$C\int_{\lambda_f}^\infty d\lambda \int_{\{x: f^*(x)>\lambda\}} u(x)\, d\mu(x) \leqslant C\int_\Omega f^*(x) u(x)\, d\mu(x) < \infty$$

and this, together with (6), shows that

$$\int_\Omega [f(x) \log^+ f(x)]\, v(x)\, d\mu(x) < \infty.$$

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