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EXAMPLES OF COMPACT NON-KÄHLER ALMOST KÄHLER MANIFOLDS

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ABSTRACT. A parametrized family of compact non-Kähler almost Kähler manifolds M^{2r+2}, $r \geq 1$, is constructed. For r odd, these manifolds are shown to have odd first Betti number, so they cannot be Kählerian.

Introduction. Few examples, compact or not, of non-Kähler almost Kähler manifolds are known. In fact, for the noncompact case only the tangent bundle (and some related tensor bundles) of nonflat Riemannian manifolds were known to admit such structure until B. Watson's report [**10**] of a large family of other noncompact examples; in 1976, W. P. Thurston [**9**] reported the existence of a compact manifold M^4, defined as a 2-torus bundle over a 2-torus, which possesses an almost Kähler structure but does not admit any Kählerian structure (an explicit description of that structure is given in [**1**]), and again B. Watson constructed more examples of this kind by considering products $(M^4 \times S^1) \times (M^4 \times S^1) \times \cdots$; as far as we know, no more examples are known in the literature.

In the present paper, a large family of compact almost Kähler manifolds M^{2r+2}, $r \geq 1$, which are non-Kähler, is constructed. Indeed, M^{2r+2} is firstly obtained as a compact complete locally affine manifold (Theorem 1.3), and this fact allows an easy computation of its first Betti number; secondly, M^{2r+2} is also described as an $(r+1)$-torus bundle over an $(r+1)$-torus, extending in this way Thurston's example to higher dimensions; and thirdly, we consider M^{2r+2} as a quotient of $H(1,r) \times S^1$ by a discrete subgroup, where $H(1,r)$ is a generalized Heisenberg group, and then we construct the almost Kähler structure on it as in Abbena's approach (which corresponds to the value $r = 1$); finally, we compute the curvature, the Ricci, the $*$-Ricci and the torsion tensors of the structure to check whether some identities, holding for a Kählerian structure, are verified here. Summing up, we have the following:

(1) For r odd, M^{2r+2} has an almost Kähler metric and, since its first Betti number is also odd (Corollary 1.4), it does not admit any Kählerian metric.

(2) For r even, M^{2r+2} has an almost Kählerian metric, and this particular metric is not Kählerian.

1. Compact quotients of the generalized Heisenberg group. The generalized Heisenberg group $H(1,r)$, $r \geq 1$, is the Lie group of real matrices of the

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form

$$\begin{pmatrix} I_r & P & T \\ 0 & 1 & Q \\ 0 & 0 & 1 \end{pmatrix}$$

where P and T both are $(r \times 1)$-column matrices, $Q \in R$. $H(1,r)$ is easily shown to be a connected and simply-connected two-step nilpotent Lie group of dimension $2r+1$.

Indeed, $H(1,r)$ is canonically diffeomorphic to R^{2r+1} through the map $\psi\colon R^{2r+1} \to H(1,r)$ given by

$$\psi(X,Y,Z) = \begin{pmatrix} I_r & -{}^tZ & X \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}, \qquad Z = (Z_1, \dots, Z_r).$$

Alternatively, $H(1,r)$ can be seen as a Lie subgroup of $A(2r+1)$, the Lie group of affine transformations of R^{2r+1}, as follows: let $G_0(1,r) \subset A(2r+1)$ be the group of affine transformations of R^{2r+1} determined by

$$G_0(1,r) = \left\{ \left(\begin{pmatrix} I_r & -{}^tD & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} B \\ C \\ D_1 \\ \vdots \\ D_r \end{pmatrix} \right), \quad \begin{array}{l} B \in R^{r\times 1} \\ D = (D_1, \dots, D_r),\ D_i \in R \\ C \in R \end{array} \right\}$$

and denote (B,C,D) a generic element in $G_0(1,r)$; then $\phi\colon G_0(1,r) \to H(1,r)$ defined by

$$\phi(B,C,D) = \begin{pmatrix} I_r & -{}^tD & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

is an isomorphism of Lie groups. In fact, the following diagram commutes:

$$\begin{array}{ccc} G_0(1,r) \times R^{2r+1} & \xrightarrow{\rho_1} & R^{2r+1} \\ \phi\times\psi\downarrow & & \downarrow\psi \\ H(1,r) \times H(1,r) & \xrightarrow{\rho_2} & H(1,r) \end{array} \tag{1.3}$$

where ρ_1 (resp. ρ_2) denotes the action of $G_0(1,r)$ onto R^{2r+1} (resp. of $H(1,r)$ onto itself).

Let $\Gamma_0 \subset G_0(1,r)$ be the Lie subgroup of $G_0(1,r)$ of those elements (B,C,D) with $B \in \mathbf{Z}^r$, $C, D_i \in \mathbf{Z}$, $i = 1,2,\dots,r$. Then, Γ_0 is a discrete uniform subgroup of $G_0(1,r)$ [**6**] and, therefore, the quotient manifold R^{2r+1}/Γ_0 is a compact complete locally affine manifold [**2**] whose fundamental group is isomorphic to Γ_0. Therefore, $\Gamma = \phi(\Gamma_0) \subset H(1,r)$ is the discrete subgroup of $H(1,r)$ of matrices with integer entries and, by virtue of (1.3), we deduce the following.

THEOREM 1.1. *Let Γ be the discrete subgroup of $H(1,r)$ of matrices with integer entries. Then the quotient manifold $M^{2r+1} = H(1,r)/\Gamma$ is a compact complete locally affine manifold whose fundamental group $\pi_1(M^{2r+1})$ is isomorphic to Γ.*

From the Hurewicz Theorem we know that $H_1(M^{2r+1}) \cong \Gamma/[\Gamma,\Gamma]$, $[\Gamma,\Gamma]$ being the commutator subgroup of Γ, hence the first Betti number $b_1(M^{2r+1})$ can be easily computed. Firstly, we remark that Γ is a non-Abelian free group of $2r+1$ generators, namely

$$\alpha_i = \begin{pmatrix} I_r & |i| & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_r & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} I_r & 0 & |i| \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $|i|$ denotes the $(r \times 1)$-column matrix with 1 at the ith row. Then, a direct computation leads to the following relations:

$$\begin{aligned} \alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1} &= 1, \\ \beta\gamma_i\beta^{-1}\gamma_i^{-1} &= 1, \\ \gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1} &= 1, \\ \alpha_i\beta\alpha_i^{-1}\beta^{-1} &= \gamma_i, \end{aligned}$$

so $\Gamma/[\Gamma,\Gamma]$ is isomorphic to the Abelian free group of $r+1$ generators, and we prove the following

THEOREM 1.2. $H_1(M^{2r+1}) \cong \bigoplus_{r+1} \mathbf{Z}$, *and hence* $b_1(M^{2r+1}) = r+1$.

Next, let us consider the product manifold $M^{2r+2} = M^{2r+1} \times S^1$; then, we can state

THEOREM 1.3. *M^{2r+2} is also a compact complete locally affine manifold; that is, there is a discrete uniform subgroup $\tilde{\Gamma}_0$ of $A(2r+2)$ such that $M^{2r+2} \cong R^{2r+2}/\tilde{\Gamma}_0$.*

PROOF. Let R be considered as Abelian Lie group, and then consider the product Lie groups $\tilde{G}_0 = G_0(1,r) \times R \subset A(2r+2)$ and $H(1,r) \times R$. Then, ϕ given by (1.2) and ψ given by (1.1) extend to $\overline{\phi}\colon \tilde{G}_0 \to H(1,r) \times R$, isomorphism of Lie groups, and to $\overline{\psi}\colon R^{2r+2} \to H(1,r) \times R$, diffeomorphism of manifolds, in such way that a commutative diagram similar to (1.3) still holds. Finally, it suffices to set $\tilde{\Gamma}_0 = \Gamma_0 \times \mathbf{Z}$ to finish the proof of the theorem.

COROLLARY 1.4. $b_1(M^{2r+2}) = r+2$.

At this point, let us remark that $M^4 = (H(1,1)/\Gamma) \times S^1$ is nothing but Thurston's famous example of a compact symplectic non-Kähler manifold, as considered by E. Abbena [**1**]. Indeed, for $r > 1$, M^{2r+2} is the natural generalization of Thurston's example to higher dimensions, since it can be seen as the bundle space of a torus bundle over a torus.

Let $\rho\colon \bigoplus_{r+1} \mathbf{Z} \to \mathrm{Diff}(T^{r+1})$ be the representation defined by

$$\rho(1,0,\ldots,0) = \mathrm{id}_{T^{r+1}},$$

$$\rho(0,\ldots,\overset{\overset{i}{\downarrow}}{1},\ldots,0) = \begin{bmatrix} 1 & \ldots & 0 \\ \vdots & a^i_{i-1} & \vdots \\ 0 & \ldots & 1 \end{bmatrix}, \qquad a^i_{i-1} = 1,$$

where "[ ]" represents the transformation of T^{r+1} covered by the linear transformation of R^{r+1}; this representation determines a bundle structure for M^{2r+2} over T^{r+1} with fiber T^{r+1}, since

$$M^{2r+2} \cong \tilde{T}^{r+1} \times_{\oplus_{r+1}\mathbf{Z}} T^{r+1},$$

where $\bigoplus_{r+1} \mathbf{Z}$ operates on $\tilde{T}^{r+1} \equiv R^{r+1}$ by covering transformations, and on T^{r+1} by ρ. Actually, in accordance with Auslander's results (mainly [**2**, Theorem 3, p. 136]), this fibered structure on M^{2r+2} is not surprising at all.

2. The examples. In order to construct an almost Kähler structure on M^{2r+2}, we first remark that there is a third way of describing the structure of the quotient manifold M^{2r+2}.

Let G be the closed connected subgroup of $\mathrm{Gl}(r+3,\mathbf{C})$ defined by

$$G = \left\{ \left. \begin{pmatrix} 1 & 0 & \ldots & 0 & a_1^{r+1} & a_1^{r+2} & 0 \\ 0 & 1 & & & & & \\ & & \ddots & & & & \cdot \\ \cdot & & & 1 & a_r^{r+1} & a_r^{r+2} & \cdot \\ \cdot & & & 0 & 1 & a_{r+1}^{r+2} & 0 \\ & & & & 0 & 1 & 0 \\ 0 & & \ldots & \cdot & 0 & 0 & e^{2\pi i a} \end{pmatrix} \right/ a_i^j, a \in R \right\}$$

that is $G = H(1,r) \times S^1$; it is not hard to check that M^{2r+2} is diffeomorphic to G/Γ, Γ being the discrete subgroup of matrices with integer entries. Next, denote by $x_i, y, z_i, t,\ 1 \le i \le r$, the coordinate functions on G defined by

$$x_i(A) = a_i^{r+1}, \quad y(A) = a_{r+1}^{r+2}, \quad z_i(A) = a_i^{r+2}, \quad t(A) = a, \quad \text{for any } A \in G.$$

Then, a standard computation leads to the following family of linearly independent left invariant 1-forms on G:

$$\alpha_i = dx_i, \quad \beta = dy, \quad \gamma_i = dz_i - x_i dy, \quad \eta = dt$$

and, since they are invariant under the action of Γ, there exist $2r+2$ 1-forms $\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma}_i, \tilde{\eta}$ on M^{2r+2} such that

$$\pi^*(\tilde{\alpha}_i) = \alpha_i, \quad \pi^*(\tilde{\beta}) = \beta, \quad \pi^*(\tilde{\gamma}_i) = \gamma_i, \quad \pi^*(\tilde{\eta}) = \eta,$$

$\tilde{\pi}\colon G \to M^{2r+2}$ being the canonical projection. Therefore, $\tilde{\alpha}_i, \tilde{\beta}, \tilde{\gamma}_i, \tilde{\eta}$ are linearly independent and globally defined on M^{2r+2}.

Now, we set

$$\tilde{F} = \tilde{\beta} \wedge \tilde{\eta} + \sum_{i=1}^{r} \tilde{\alpha}_i \wedge \tilde{\gamma}_i.$$

$\tilde{F}$ is a 2-form of maximal rank and, since $\pi^*(d\tilde{F}) = d(\pi^*\tilde{F}) = 0$, it is closed; hence $\tilde{F}$ is a symplectic form on M^{2r+2}. Actually, there exist many metrics and almost complex structures on M^{2r+2} whose Kähler form is $\tilde{F}$. In the sequel, we shall write down such an almost Hermitian structure on M^{2r+2}.

First, we note that the dual fields of $\{\alpha_i, \beta, \gamma_i, \eta\}$ on G are, respectively,

$$X_i = \frac{\partial}{\partial x_i}, \quad Y = \frac{\partial}{\partial y} + \sum_{i=1}^{r} x_i \frac{\partial}{\partial z_i}, \quad Z_i = \frac{\partial}{\partial z_i}, \quad T = \frac{\partial}{\partial t}$$

and they form an orthonormal frame field with respect to the left invariant metric on G defined by

$$ds^2 = \sum_{i=1}^{r} (\alpha_i^2 + \gamma_i^2) + \beta^2 + \eta^2.$$

Let J be the tensor field of type $(1,1)$ on G defined by

$$JX_i = Z_i, \quad JZ_i = -X_i, \quad JY = T, \quad JT = -Y.$$

Then J is a left invariant almost complex structure on G, and ds^2 above is an Hermitian metric with respect to J. Moreover, both J and ds^2 are, in particular, invariant under the action of Γ, so both project down to M^{2r+2}; therefore, there exists an almost complex structure $\tilde{J}$ on M^{2r+2} such that $\pi_* JX = \tilde{J}\pi_* X$ for any $X \in g$, where g denotes the Lie algebra of G, and the corresponding induced Hermitian metric is given by

$$d\tilde{s}^2 = \sum_{i=1}^{r} (\tilde{\alpha}_i^2 + \tilde{\gamma}_i^2) + \tilde{\beta}^2 + \tilde{\eta}^2.$$

Since $\tilde{F}$ is the Kähler form of this almost Hermitian structure and it is closed, it follows that M^{2r+2} is an almost Kähler manifold. In fact, we shall prove that such structure on M^{2r+2} is non-Kähler; to be more precise, we can state

THEOREM 2.1. *The almost Hermitian structure $(\tilde{J}, d\tilde{s}^2)$ on M^{2r+2} is strictly almost Kähler. In fact, no Kähler structures exist on M^{2r+2} for odd r.*

PROOF. From Corollary 1.4 we know that $b_1(M^{2r+2}) = r+2$; hence, no Kähler structures could exist on M^{2r+2} when r is odd, because of the well-known fact that the first Betti number of a compact Kähler manifold must be even. But when r is even we obtain no topological obstructions in this way, so we are obliged to proceed through a different way; in fact, we shall compute the curvature tensor field of M^{2r+2}, for any r, and the computations will be done in the Lie algebra g of G because M^{2r+2} and G are locally isometric.

So, let $\langle\ ,\ \rangle$ denote the metric ds^2 on G; then, its Riemannian connection ∇ is given by

$$\begin{aligned} 2\langle \nabla_U V, W\rangle = {} & U\langle V, W\rangle + V\langle U, W\rangle - W\langle U, V\rangle \\ & - \langle U, [V,W]\rangle - \langle V, [U,W]\rangle + \langle W, [U,V]\rangle \end{aligned}$$

for arbitrary $U, V, W \in g$; therefore, for the covariant derivatives of the frame fields $X_i, Y, Z_i, T \in g$ one obtains

$$\nabla_{X_i} Z_i = \nabla_{Z_i} X_i = -\tfrac{1}{2} Y, \quad \nabla_{X_i} Y = -\nabla_Y X_i = \tfrac{1}{2} Z_i, \quad \nabla_{Z_i} Y = \nabla_Y Z_i = \tfrac{1}{2} X_i,$$

the other covariant derivatives are all zero. And, for the curvature tensor R given by $R(U,V,W,Q) = \langle R(U,V)W,Q\rangle$, we get

$$\begin{aligned} R(X_i,X_j,Z_i,Z_j) &= \tfrac{1}{4}, \qquad i \neq j,\\ R(X_i,Z_j,X_j,Z_i) &= -\tfrac{1}{4},\\ R(X_i,Y,X_i,Y) &= \tfrac{3}{4},\\ R(Z_i,Y,Z_i,Y) &= -\tfrac{1}{4}, \end{aligned}$$

the other components being zero. Therefore,

$$\begin{aligned} R(X_1,X_r,Z_1,Z_r) &= \tfrac{1}{4},\\ R(X_1,X_r,JZ_1,JZ_r) &= R(X_1,X_r,X_1,X_r) = 0, \end{aligned}$$

and thus $R(X_1,X_r,Z_1,Z_r) \neq R(X_1,X_r,JZ_1,JZ_r)$; hence, the structure on M^{2r+2} is not Kählerian.

Another proof of the fact that the structure we are considering on M^{2r+2} is not Kähler can be obtained by computing the Ricci and the $*$-Ricci tensors of $(M^{2r+2}, \tilde{J}, d\tilde{s}^2)$; once more, we do the computation in g.

The Ricci tensor ρ is given by

$$\begin{aligned} \rho(U,V) = \sum_{i=1}^{r} &\{R(U,X_i,V,X_i) + R(U,Z_i,V,Z_i)\}\\ &+ R(U,Y,V,Y) + R(U,T,V,T) \end{aligned}$$

for any $U,V \in g$; then, one finds easily that the nonvanishing components of ρ with respect to the basis $\{X_i,Y,Z_i,T\}$ of g are

$$\rho(X_i,X_i) = \tfrac{1}{2}, \quad \rho(Z_i,Z_i) = -\tfrac{1}{2}, \quad \rho(Y,Y) = \tfrac{1}{2}r.$$

Similarly, the $*$-Ricci tensor ρ^* is given by

$$\begin{aligned} \rho^*(U,V) = \sum_{i=1}^{r} &\{R(U,X_i,JV,JX_i) + R(U,Z_i,JV,JZ_i)\}\\ &+ R(U,Y,JV,JY) + R(U,T,JV,JT) \end{aligned}$$

and a simple computation shows that $\rho^* = 0$.

Therefore, $\rho \neq \rho^*$, while in a Kählerian manifold both tensors must be equal.

Finally, the nonintegrability of the almost complex structure J on G (and hence of $\tilde{J}$ on M^{2r+2}) can be easily checked by computing the components of its Nijenhuis tensor N_J; one finds easily that the nonvanishing components of N_J are

$$\begin{aligned} N_J(X_i,Y) &= -N_J(Z_i,T) = -Z_i,\\ N_J(X_i,T) &= N_J(Z_i,Y) = -X_i, \end{aligned}$$

and so $(M^{2r+2}, \tilde{J}, d\tilde{s}^2)$ is a strictly almost Kähler manifold (in the terminology of Watson's paper).

REMARK. Obviously, the almost Kähler structure defined on M^{2r+2} is not the unique possible of this type on the manifold. For example, for $r = 1$, the structure which we have defined on M^4 is not the same as that of Abbena's paper [**1**], but both, Abbena's and ours, can be easily shown to be equivalent; in fact, one can easily define an automorphism of G preserving the almost complex structures and which projects down to M^{2r+2}.

ADDED IN PROOF. After this paper was completed, L. A. Cordero, M. Fernández and Alfred Gray (*Symplectic manifolds with no Kähler structure*, preprint) proved that M^{2r+2} can have no Kähler structures for any $r \geq 1$.

REFERENCES

1. E. Abbena, *An example of an almost Kähler manifold which is not Kählerian*, Boll. Un. Mat. Ital. (6) **3A** (1984), 383–392.
2. L. Auslander, *The structure of complete locally affine manifolds*, Topology **3** suppl. 1 (1964), 131–139.
3. S. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
4. A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. **28** (1976), 601–612.
5. ——, *Homogeneous almost Hermitian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino (to appear).
6. Y. Haraguchi, *Sur une généralisation des structures de contact*, Thèse d'Université, Univ. de Haute Alsace, Mulhouse, 1981.
7. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vols. I, II, Interscience, New York, 1963–1969.
8. J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), 293–329.
9. W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467–468.
10. B. Watson, *New examples of strictly almost Káhler manifolds*, Proc. Amer. Math. Soc. **88** (1983), 541–544.

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