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SLLN AND CONVERGENCE RATES FOR NEARLY ORTHOGONAL SEQUENCES OF RANDOM VARIABLES[1]

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ABSTRACT. Let $\{X_k: k \geqslant 1\}$ be a sequence of random variables with finite second moments $EX_k^2 = \sigma_k^2 < \infty$ for which $|EX_k X_l| \leqslant \sigma_k \sigma_l \rho(|k-l|)$, where $\{\rho(j): j \geqslant 0\}$ is a sequence of nonnegative numbers with $\sum_{j=0}^{\infty} \rho(j) < \infty$. In particular, in the case of orthogonality, $\rho(j) = 0$ for $j \geqslant 1$. We prove strong laws for the first arithmetic means $\zeta_n = n^{-1}\sum_{k=1}^{n} X_k$ and the Cesàro means

$$\tau_n = n^{-1} \sum_{k=1}^{n} \left(1 - (k-1)n^{-1}\right) X_k.$$

The convergence rates are studied in the form $P\{\sup_{n>2^p} |\zeta_n| > \varepsilon\}$ and $P\{\sup_{n>2^p} |\tau_n| > \varepsilon\}$, where $\varepsilon > 0$ is fixed and p tends to ∞. At the end, the case where $\sum_{j=0}^{\infty} \rho(j) = \infty$ is also treated.

1. Introduction. Let $\{X_k: k \geqslant 1\}$ be an orthogonal sequence of random variables (rv's), i.e.

$$EX_k X_l = 0 \qquad (k \neq l;\ k, l \geqslant 1) \tag{1.1}$$

with finite second moments

$$EX_k^2 = \sigma_k^2 \qquad (k \geqslant 1). \tag{1.2}$$

We will consider the first arithmetic means $\zeta_n = (1/n)\sum_{k=1}^{n} X_k$ as well as the Cesàro means (of order 1)

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) X_k \qquad (n \geqslant 1).$$

A consequence of the Rademacher-Menshov theorem, well known in the theory of orthogonal series, is the following (see e.g. [**3**, pp. 86, 87]).

THEOREM A. *If $\{X_k\}$ is an orthogonal sequence of rv's with* (1.2) *and*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2 < \infty, \tag{1.3}$$

then

$$\lim_{n \to \infty} \zeta_n = 0 \quad a.s. \tag{1.4}$$

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In this paper the logarithms are to the base 2.

It is also pointed out that the sufficient condition (1.3) is the best possible (see Tandori [**4**]).

The next theorem is due to the author [**2**].

THEOREM B. *If* $\{X_k\}$ *is an orthogonal sequence of rv's with* (1.2) *and*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty, \tag{1.5}$$

then

$$\lim_{n\to\infty} \tau_n = 0 \quad a.s. \tag{1.6}$$

2. Main results. The orthogonality condition (1.1) can be weakened in Theorems A and B maintaining conclusions (1.4) and (1.6), respectively.

To be more precise, we say that the sequence $\{X_k\}$ of rv's satisfying condition (1.2) is *quasi-orthogonal* (or *nearly orthogonal*) if there exists a sequence $\{\rho(j): j \geqslant 0\}$ of nonnegative numbers such that

$$|EX_kX_l| \leqslant \sigma_k\sigma_l\rho(|k-l|) \qquad (k, l \geqslant 1) \tag{2.1}$$

and

$$\sum_{j=0}^{\infty} \rho(j) < \infty. \tag{2.2}$$

If $EX_k = 0$ $(k \geqslant 1)$, then (2.1) is equivalent to

$$|\mathrm{Corr}(X_k, X_l)| \leqslant \rho(|k-l|) \qquad (k, l \geqslant 1).$$

Also, we may assume that $\rho(0) = 1$ and $0 \leqslant \rho(j) \leqslant 1$ $(j \geqslant 1)$.

Now, the generalizations of Theorems A and B are the following:

THEOREM 1. *If* $\{X_k\}$ *is a quasi-orthogonal sequence of rv's, then* (1.3) *implies* (1.4).

THEOREM 2. *If* $\{X_k\}$ *is a quasi-orthogonal sequence of rv's, then* (1.5) *implies* (1.6).

Both theorems will be obtained as corollaries of the next two theorems stating convergence rates.

THEOREM 3. *If* $\{X_k\}$ *is a quasi-orthogonal sequence of rv's, then* (1.3) *implies, for every* $\varepsilon > 0$,

(2.3)

$$P\left\{\sup_{n>2^p} |\zeta_n| > \varepsilon\right\} = O\left\{\frac{1}{2^{2p}}\sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}[\log(k+1)]^2\right\} \qquad (p \geqslant 0).$$

THEOREM 4. *If* $\{X_k\}$ *is a quasi-orthogonal sequence of rv's, then* (1.5) *implies, for every* $\varepsilon > 0$,

$$P\left\{\sup_{n>2^p} |\tau_n| > \varepsilon\right\} = O\left\{\frac{1}{2^{2p}}\sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}\right\} \qquad (p \geqslant 0). \tag{2.4}$$

We note two other consequences of Theorems 3 and 4.

COROLLARY 1. *Assume $\{X_k\}$ is a quasi-orthogonal sequence of rv's and*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2}[\log(k+1)]^2\lambda(k) < \infty, \tag{2.5}$$

where $\{\lambda(k)\colon k \geqslant 1\}$ is a nondecreasing sequence of positive numbers such that the sequence $\{k^2/\lambda(k)\colon k \geqslant 1\}$ is also nondecreasing and tends to ∞. Then, for every $\varepsilon > 0$,

$$\lim_{p\to\infty} \lambda(2^p) P\left\{ \sup_{n>2^p} |\zeta_n| > \varepsilon \right\} = 0. \tag{2.6}$$

COROLLARY 2. *Assume $\{X_k\}$ is a quasi-orthogonal sequence of rv's and*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2}\lambda(k) < \infty,$$

where $\{\lambda(k)\}$ is the same as in Corollary 1. Then, for every $\varepsilon > 0$,

$$\lim_{p\to\infty} \lambda(2^p) P\left\{ \sup_{n>2^p} |\tau_n| > \varepsilon \right\} = 0.$$

We briefly indicate, e.g., how Corollary 1 can be deduced from Theorem 3.

It follows from (2.5) that (even dropping the factor $[\log(k+1)]^2$), via the Kronecker lemma (see e.g. [**3**, p. 35]),

$$\lim_{p\to\infty} \frac{\lambda(2^p)}{2^{2p}} \sum_{k=1}^{2^p} \sigma_k^2 = 0.$$

On the other hand, again by (2.5),

$$\begin{aligned}\lim_{p\to\infty} \lambda(2^p) &\sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}[\log(k+1)]^2 \\ &\leqslant \lim_{p\to\infty} \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}[\log(k+1)]^2\lambda(k) = 0.\end{aligned}$$

That is, (2.3) implies (2.6).

3. Auxiliary results.

LEMMA 1. *If $\{X_k\}$ is a sequence of rv's satisfying conditions* (1.2) *and* (2.1), *then*

$$E\left[\sum_{k=a+1}^{a+n} X_k\right]^2 \leqslant \left(1 + 2\sum_{j=1}^{n-1}\rho(j)\right) \sum_{k=a+1}^{a+n} \sigma_k^2 \qquad (a \geqslant 0, n \geqslant 1). \tag{3.1}$$

PROOF. Squaring out, using (1.2) and (2.1), letting $j = l - k$ and, finally, applying the Cauchy inequality yields (3.1):

$$\begin{aligned} E\left[\sum_{k=a+1}^{a+n} X_k\right]^2 &= \sum_{k=a+1}^{a+n} EX_k^2 + 2\sum_{k=a+1}^{a+n-1}\sum_{l=k+1}^{a+n} EX_kX_l \\ &\leqslant \sum_{k=a+1}^{a+n} \sigma_k^2 + 2\sum_{k=a+1}^{a+n-1}\sum_{l=k+1}^{a+n} \sigma_k\sigma_l\rho(l-k) \\ &= \sum_{k=a+1}^{a+n} \sigma_k^2 + 2\sum_{j=1}^{n-1}\rho(j)\sum_{k=a+1}^{a+n-j} \sigma_k\sigma_{k+j} \\ &\leqslant \left(1 + 2\sum_{j=1}^{n-1}\rho(j)\right)\sum_{k=a+1}^{a+n}\sigma_k^2, \end{aligned}$$

since

$$\begin{aligned} \sum_{k=a+1}^{a+n-j}\sigma_k\sigma_{k+j} &\leqslant \left\{\sum_{k=a+1}^{a+n-j}\sigma_k^2 \sum_{k=a+1}^{a+n-j}\sigma_{k+j}^2\right\}^{1/2} \\ &= \left\{\sum_{k=a+1}^{a+n-j}\sigma_k^2 \sum_{k=a+j+1}^{a+n}\sigma_k^2\right\}^{1/2} \leqslant \sum_{k=a+1}^{a+n}\sigma_k^2. \end{aligned}$$

In the proof of Theorem 4 we need a slightly more general form of Lemma 1.

LEMMA 2. *If* $\{X_k\}$ *is a sequence of rv's satisfying conditions* (1.2) *and* (2.1), *and* $\{b_k\colon k \geqslant 1\}$ *is a sequence of numbers, then*

$$E\left[\sum_{k=a+1}^{a+n} b_kX_k\right]^2 \leqslant \left(1 + 2\sum_{j=1}^{n-1}\rho(j)\right)\sum_{k=a+1}^{a+n} b_k^2\sigma_k^2 \qquad (a \geqslant 0, n \geqslant 1). \tag{3.2}$$

Indeed, applying Lemma 1 for $\{Y_k = b_kX_k\}$, we get immediately Lemma 2.

The next lemma is a special case of the maximal inequality in [**1**, Theorem 3].

LEMMA 3. *If* $\{X_k\}$ *is a sequence of rv's such that condition* (3.1) *is satisfied, then*

$$E\left[\max_{1\leqslant m\leqslant n}\left|\sum_{k=a+1}^{a+m} X_k\right|\right]^2 \leqslant [\log 2n]^2\left(1 + 2\sum_{j=1}^{n-1}\rho(j)\right)\sum_{k=a+1}^{a+n}\sigma_k^2 \qquad (a \geqslant 0, n \geqslant 1). \tag{3.3}$$

4. Proofs of Theorems 3 and 4.

PROOF OF THEOREM 3. Obviously,

$$P\left\{\sup_{n>2^p}|\zeta_n| > \varepsilon\right\} \leqslant \sum_{q=p}^{\infty} P\left\{\max_{2^q<n\leqslant 2^{q+1}}|\zeta_n| > \varepsilon\right\}. \tag{4.1}$$

A simple estimate shows

$$\max_{2^q<n\leqslant 2^{q+1}}|\zeta_n| \leqslant |\zeta_{2^q}| + \frac{1}{2^q}\max_{2^q<n\leqslant 2^{q+1}}\left|\sum_{k=2^q+1}^{n} X_k\right|.$$

On one hand, by (3.1) and (2.2),

$$E\zeta_{2^q}^2 = \frac{O\{1\}}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2. \tag{4.2}$$

On the other hand, by (3.3) and (2.2),

$$\begin{aligned} E\left[\max_{2^q < n \leqslant 2^{q+1}} \left| \sum_{k=2^q+1}^{n} X_k \right|\right]^2 &= O\{1\}\left[\log 2^{q+1}\right]^2 \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 \\ &= O\{1\} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 [\log(k+1)]^2. \end{aligned}$$

Thus, by the Chebyshev inequality,

$$\begin{aligned} P\left\{\max_{2^q < n \leqslant 2^{q+1}} |\zeta_n| > \varepsilon\right\} &\leqslant P\left\{|\zeta_{2^q}| > \frac{\varepsilon}{2}\right\} + P\left\{\max_{2^q < n \leqslant 2^{q+1}} \left|\sum_{k=2^q+1}^{n} X_k\right| > \varepsilon 2^{q-1}\right\} \\ &= \frac{O\{1\}}{\varepsilon^2}\left(\frac{1}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2 + \frac{1}{2^{2q}} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 [\log(k+1)]^2\right). \end{aligned} \tag{4.3}$$

Keeping (4.1) in mind, simple calculations show

$$\begin{aligned} \sum_{q=p}^{\infty} \frac{1}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2 &= \sum_{k=1}^{2^p} \sigma_k^2 \sum_{q=p}^{\infty} \frac{1}{2^{2q}} + \sum_{k=2^p+1}^{\infty} \sigma_k^2 \sum_{q:\, 2^q \geqslant k} \frac{1}{2^{2q}} \\ &\leqslant \frac{4}{3}\left(\frac{1}{2^{2p}} \sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}\right) \end{aligned} \tag{4.4}$$

and

$$\sum_{q=p}^{\infty} \frac{1}{2^{2q}} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 [\log(k+1)]^2 \leqslant 4 \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2. \tag{4.5}$$

Collecting (4.1) and (4.3)–(4.5) yields (2.3).

PROOF OF THEOREM 4. Similarly to (4.1),

$$P\left\{\sup_{n > 2^p} |\tau_n| > \varepsilon\right\} \leqslant \sum_{q=p}^{\infty} P\left\{\max_{2^q < n \leqslant 2^{q+1}} |\tau_n| > \varepsilon\right\}. \tag{4.6}$$

This time we avoid using the maximal inequality (3.3). Instead, we estimate as follows:

$$\max_{2^q < n \leqslant 2^{q+1}} |\tau_n| \leqslant |\zeta_{2^q}| + |\tau_{2^q} - \zeta_{2^q}| + \max_{2^q < n \leqslant 2^{q+1}} |\tau_n - \tau_{2^q}|,$$

whence

$$\begin{aligned} P\left\{\max_{2^q < n \leqslant 2^{q+1}} |\tau_n| > \varepsilon\right\} &\leqslant P\left\{|\zeta_{2^q}| > \frac{\varepsilon}{3}\right\} + P\left\{|\tau_{2^q} - \zeta_{2^q}| > \frac{\varepsilon}{3}\right\} \\ &\quad + P\left\{\max_{2^q < n \leqslant 2^{q+1}} |\tau_n - \tau_{2^q}| > \frac{\varepsilon}{3}\right\}. \end{aligned} \tag{4.7}$$

According to this, we complete the proof in three steps:

(i) By (4.2) and (4.4),

$$(4.8)\quad \sum_{q=p}^{\infty} P\left\{|\zeta_{2^q}| > \frac{\varepsilon}{3}\right\} \leqslant \frac{9}{\varepsilon^2}\sum_{q=p}^{\infty} E\zeta_{2^q}^2 = \frac{O\{1\}}{\varepsilon^2}\left(\frac{1}{2^{2p}}\sum_{k=1}^{2^p}\sigma_k^2 + \sum_{k=2^p+1}^{\infty}\frac{\sigma_k^2}{k^2}\right).$$

(ii) Taking into account the representation

$$\tau_{2^q} - \zeta_{2^q} = -\frac{1}{2^{2q}}\sum_{k=2}^{2^q}(k-1)X_k,$$

via (3.2), (2.2) and (4.2),

$$E[\tau_{2^q} - \zeta_{2^q}]^2 = \frac{O\{1\}}{2^{4q}}\sum_{k=2}^{2^q}(k-1)^2\sigma_k^2 = \frac{O\{1\}}{2^{2q}}\sum_{k=2}^{2^q}\sigma_k^2 = O\{1\}E\zeta_{2^q}^2.$$

By this and (4.8),

$$(4.9)\quad \sum_{q=p}^{\infty} P\left\{|\tau_{2^q} - \zeta_{2^q}| > \frac{\varepsilon}{3}\right\} \leqslant \frac{9}{\varepsilon^2}\sum_{q=p}^{\infty} E[\tau_{2^q} - \zeta_{2^q}]^2$$
$$= \frac{O\{1\}}{\varepsilon^2}\left(\frac{1}{2^{2p}}\sum_{k=1}^{2^p}\sigma_k^2 + \sum_{k=2^p+1}^{\infty}\frac{\sigma_k^2}{k^2}\right).$$

(iii) By the Cauchy inequality,

$$(4.10)\quad \max_{2^q < n \leqslant 2^{q+1}} |\tau_n - \tau_{2^q}| \leqslant \sum_{n=2^q+1}^{2^{q+1}} |\tau_n - \tau_{n-1}| \leqslant \left\{\sum_{n=2^q+1}^{2^{q+1}} n[\tau_n - \tau_{n-1}]^2\right\}^{1/2}.$$

The representation

$$\tau_n - \tau_{n-1} = \sum_{k=1}^{n}\left[\frac{(k-1)(2n-1)}{n^2(n-1)^2} - \frac{1}{n(n-1)}\right]X_k$$

can be easily checked, whence, via (3.2) and (2.2),

$$E[\tau_n - \tau_{n-1}]^2 = \frac{O\{1\}}{n^2(n-1)^2}\sum_{k=1}^{n}\sigma_k^2.$$

Thus, by (4.2),

$$\sum_{n=2^q+1}^{2^{q+1}} nE[\tau_n - \tau_{n-1}]^2 = O\{1\}\sum_{n=2^q+1}^{2^{q+1}}\frac{1}{n(n-1)^2}\sum_{k=1}^{n}\sigma_k^2$$
$$= \frac{O\{1\}}{2^{2q}}\sum_{k=1}^{2^{q+1}}\sigma_k^2 = O\{1\}E\zeta_{2^{q+1}}^2.$$

By this, (4.10) and (4.4),

$$(4.11)\quad \sum_{q=p}^{\infty} P\left\{\max_{2^q < n \leqslant 2^{q+1}} |\tau_n - \tau_{2^q}| > \frac{\varepsilon}{3}\right\} = \frac{O\{1\}}{\varepsilon^2}\sum_{q=p}^{\infty} E\zeta_{2^{q+1}}^2$$
$$= \frac{O\{1\}}{\varepsilon^2}\left\{\frac{1}{2^{2p+2}}\sum_{k=1}^{2^{p+1}}\sigma_k^2 + \sum_{k=2^{p+1}+1}^{\infty}\frac{\sigma_k^2}{k^2}\right\}.$$

Putting together (4.6)–(4.9) and (4.11) gives (2.4).

5. The case when $\{X_k\}$ is not nearly orthogonal. In this concluding section we assume that conditions (1.2) and (2.1) are satisfied, but (2.2) is not, i.e.

$$\sum_{j=0}^{\infty} \rho(j) = \infty. \tag{5.1}$$

We will require, for the sake of simplicity in calculations, that the sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \rho(j) : n \geqslant n_0 \right\} \text{ is nonincreasing from some } n_0 \text{ on.} \tag{5.2}$$

It is easy to see that this requirement is equivalent to

$$\rho(n) \leqslant \frac{1}{n} \sum_{j=0}^{n-1} \rho(j) \qquad (n \geqslant n_0).$$

The following three theorems can be proved by using methods similar to those in §4, and using Lemmas 4 and 5 below.

THEOREM 5. *If $\{X_k\}$ is a sequence of rv's satisfying* (1.2), (2.1) *and* (5.2), *then the condition*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \left(\sum_{j=0}^{k-1} \rho(j) \right) [\log(k+1)]^2 < \infty \qquad (\rho(0) = 1) \tag{5.3}$$

implies (1.4).

If the divergence in (5.1) is "fast enough" in the sense that there exist a number $r > 1$ and a positive integer p_0 such that

$$\frac{\sum_{j=0}^{2^{p+1}-1} \rho(j)}{\sum_{j=0}^{2^p-1} \rho(j)} \geqslant r \qquad (p \geqslant p_0;\ \rho(0) = 1), \tag{5.4}$$

then the factor $[\log(k+1)]^2$ in (5.3) becomes superfluous.

THEOREM 6. *If $\{X_k\}$ is a sequence of rv's satisfying* (1.2), (2.1), (5.2) *and* (5.4), *then the condition*

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \left(\sum_{j=0}^{k-1} \rho(j) \right) < \infty \tag{5.5}$$

implies (1.4).

THEOREM 7. *If $\{X_k\}$ is a sequence of rv's satisfying* (1.2), (2.1) *and* (5.2), *then condition* (5.5) *implies* (1.6).

Here we present only two lemmas, without entering into details. The first of them is a special case of the maximal inequality [**1**, Theorem 4].

LEMMA 4. *If $\{X_k\}$ is a sequence of rv's such that conditions* (3.1) *and* (5.4) *are satisfied, then*

$$E\left[\max_{1 \leqslant m \leqslant n} \left| \sum_{k=a+1}^{a+m} X_k \right| \right]^2 = O\left\{ \sum_{j=0}^{n-1} \rho(j) \right\} \sum_{k=a+1}^{a+n} \sigma_k^2 \qquad (a \geqslant 0, n \geqslant 1).$$

This lemma is crucial in the proof of Theorem 6.

The second lemma is concerned with numerical series and can be easily checked.

LEMMA 5. *If condition* (5.2) *is satisfied, then*

$$\sum_{p:\,2^p \geqslant k} \frac{1}{2^{2\alpha p}} \sum_{j=0}^{2^p-1} \rho(j) = O\left\{\frac{1}{k^{2\alpha}} \sum_{j=0}^{k-1} \rho(j)\right\} \qquad (\alpha = 1 \textit{ and } 2)$$

and

$$\sum_{n=k}^{\infty} \frac{1}{n^3} \sum_{j=0}^{n-1} \rho(j) = O\left\{\frac{1}{k^2} \sum_{j=0}^{k} \rho(j)\right\} \qquad (k \geqslant 1).$$

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