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SOME ELEMENTS IN THE STABLE HOMOTOPY OF SPHERES

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ABSTRACT. An infinite family in ${}_2\pi_*^s$ at Adams filtration 4 is constructed.

Let A denote the mod 2 Steenrod algebra. There is a spectral sequence $\{E_r^{s,t}\}$ which converges to the 2-primary component of the stable homotopy groups of spheres ${}_2\pi_*^s$ and has

$$E_2^{s,t} \cong \operatorname{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2).$$

This is known as the mod 2 Adams spectral sequence [**1**].

Let $h_i \in \operatorname{Ext}_A^{1,2^i}(\mathbf{Z}_2, \mathbf{Z}_2)$ be the class corresponding to the generator $\mathrm{Sq}^{2^i} \in A$ as described by Adams in [**2**]. J. P. May shows in [**12**] that $\operatorname{Ext}_A^{3,22}(\mathbf{Z}_2, \mathbf{Z}_2) \cong \mathbf{Z}_2$ generated by an element called c_1. D. M. Davis shows in [**9**] that $c_1 h_i \neq 0$ in $\operatorname{Ext}_A^{4,2^i+11}(\mathbf{Z}_2, \mathbf{Z}_2)$ for $i \geqslant 11$. In this paper we prove

THEOREM 1. *For $i \geqslant 11$, $c_1 h_i$ survives the Adams spectral sequence and so detects homotopy elements in $\pi_{2^i+18}^s$.*

This establishes an infinite family in ${}_2\pi_*^s$ at Adams filtration 4. This family cannot be constructed from elements of lower Adams filtrations so far exposed [**7, 11**].

Using May spectral sequence [**12**] one can show that $c_1 h_i \neq 0$ for $5 \leqslant i \leqslant 10$ also, and our result is true for these cases too. The case $i = 5$ was published by M. C. Tangora in [**14**] and other cases have not yet formally published.

The proof is based on Mahowald's technique in [**11**], where he proves $h_1 h_i$ detects homotopy elements for $i \geqslant 3$. For $a, b \in \mathbf{Z}$ with $b < a$ there is a spectrum P_b^a which, when $b > 0$, is the suspension spectrum of stunted real projective space $\mathbf{R}P^a/\mathbf{R}P^{b-1}$. These can be defined as Thom spectra or using James periodicity as in [**3**]. Let S^n denote the sphere spectrum in stable dimension n. In the following, the cohomology groups have mod 2 coefficients.

PROPOSITION 2. *Suppose $i \geqslant 1$. Then*

(a) *There is a map $f\colon P_{-2^{i-1}-1}^{-2} \to S^{-2^i-1}$ such that, in the mapping cone $X = S^{-2^i-1} \cup_f CP_{-2^{i-1}-1}^{-2}$, the Steenrod operation*

$$\mathrm{Sq}^{2^i}\colon H^{-2^i-1}(X) = \mathbf{Z}_2 \to H^{-1}(X) = \mathbf{Z}_2$$

is nonzero.

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(b) *There is a map* $g\colon S^{17} \to P^{-2}_{-2^{i-1}-1}$ *such that the composite*

$$S^{17} \xrightarrow{g} P^{-2}_{-2^{i-1}-1} \xrightarrow{p} S^{-2}$$

is detected by c_1 *where* p *is the collapsing map.*

This implies that $c_1 h_i$ are permanent cycles in the Adams spectral sequence (in deducing this, one needs the fact that in X primary operations from the bottom cell to other cells of dimensions $\leqslant -2^{i-1} - 1$ are trivial and this follows by dimensional reasons). Since $\mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2) = 0$ for $s \leqslant 2$ and $t - s = 2^i + 19$ [**2**], $c_1 h_i$ are not boundaries. This proves Theorem 1.

The result 2(a) is not new; it is primarily due to Mahowald [**11**]. A detailed proof of it can be found in [**8**]. For completeness we sketch the proof as follows.

Consider the double loop space $\Omega^2 S^3$ where S^3 now stands for the 3-sphere. V. P. Snaith shows in [**13**] that there is a stable splitting

$$\Omega^2 S^3_+ \simeq \bigvee_{k \geqslant 0} D_k.$$

Here $\Omega^2 S^3_+$ denotes $\Omega^2 S^3$ with a disjoint base point, $D_0 = S^0$ (two points) and for $k \geqslant 1$

$$D_k = F(\mathbf{R}^2, k)_+ \wedge_{\Sigma_k} \big(\underbrace{S^1 \wedge \cdots \wedge S^1}_{k}\big),$$

where $F(\mathbf{R}^2, k)$ is the configuration space of k-tuples of distinct points in $\mathbf{R}^2$. In [**11**] Mahowald showed that for $k \geqslant 1$

(1) $H^*(D_k) \cong A/A\{\chi(\mathrm{Sq}^i) \colon i > [k/2]\}$ generated by a class $u_k \in H^k(D_k)$,

and conjectured that, at prime 2, D_k had the stable homotopy type of the Brown-Gitler spectrum $B([k/2])$ [**5**]. Here $\chi\colon A \to A$ is the canonical antiautomorphism of A. Mahowald's conjecture was later proved by Brown and Peterson in [**6**]. Brown-Gitler spectra $B([k/2])$ are characterized by (1) together with an additional property which will not be described here (see [**6**]). A corollary of this property is the following:

(2) Let M be a closed manifold of dimension $n \leqslant k$ and let $T(\nu_M)$ be the Thom spectrum of the stable normal bundle ν_M of M with Thom class $U \in H^*(T(\nu_M))$ lying in dimension zero. Then $\exists$ a map $\phi\colon \Sigma^k T(\nu_M) \to D_k$ such that $\phi^*(u_k) = U$.

Applying (2) to $M = \mathbf{R}P^{2^{i-1}}$ and letting $\bar{\nu} = \nu_{\mathbf{R}P^{2^{i-1}}}$ we have the following:

(3) $\exists$ a map $\phi\colon \Sigma^{2^{i-1}} T(\bar{\nu}) \to D_{2^{i-1}}$ such that $\phi^*(u_{2^{i-1}}) = U$.

We have $\Sigma^{2^{i-1}} T(\bar{\nu}) \simeq \Sigma^{2^i+1} P^{-2}_{-2^{i-1}-1}$ [**4, 10**]. Since the Stiefel-Whitney class $\omega_{2^{i-1}-1}(\bar{\nu})$ is nonzero, it follows that $\mathrm{Sq}^{2^{i-1}-1}(U) \neq 0$, which is the top class of $H^*(\Sigma^{2^i+1} P^{-2}_{-2^{i-1}-1})$. Let $v = \mathrm{Sq}^{2^{i-1}-1}(u_{2^{i-1}}) \in H^{2^i-1}(D_{2^{i-1}}) = \mathbf{Z}_2$. From (3) we see that

(4) $\exists$ a map $\phi\colon \Sigma^{2^i+1} P^{-2}_{-2^{i-1}-1} \to D_{2^{i-1}}$ such that $\phi^*(v) \neq 0$ in $H^{2^i-1}(\Sigma^{2^i+1} P^{-2}_{-2^{i-1}-2}) = \mathbf{Z}_2$.

Mahowald also shows in **[11]**, and this is crucial, that

(5) $\exists$ a stable map $\tilde{\eta}\colon \Omega^2 S^3_+ \simeq \bigvee_{k\geqslant 0} D_k \to S^0$ such that
$\mathrm{Sq}^{2^i}\colon H^0\big(S^0 \cup_{\tilde{\eta}|D_{2^i-1}} CD_{2^i-1}\big) = \mathbf{Z}_2 \to H^{2^i}\big(S^0 \cup_{\tilde{\eta}|D_{2^i-1}} CD_{2^i-1}\big) = \mathbf{Z}_2$
is nonzero for each $i \geqslant 1$.

Proposition 2(a) follows by (4) and (5).

We remark that in **[11]** Mahowald only considers $\Omega^2 S^9$ to get a result analogous to (5). But his argument also works for $\Omega^2 S^3$.

Our main work is to show Proposition 2(b). Let η, ν and σ be the Hopf classes in π_1^s, π_3^s and π_7^s, respectively **[15]**. H. Toda has shown **[15]** that $\langle \nu, \eta, \sigma^2\rangle$ consists of a single element $\bar{\sigma}$ which has order 2. May observed **[12]** that $\bar{\sigma}$ is detected by c_1. From the definition of the Toda bracket $\langle \nu, \eta, \sigma^2\rangle$ we see that, for any extension $\tilde{\nu}$: $S^1 \cup_\eta e^3 \to S^{-2}$ of $\nu\colon S^1 \to S^{-2}$ and any coextension $\overline{\sigma^2}\colon S^{17} \to S^1 \cup_\eta e^3$ of σ^2: $S^{17} \to S^3$,

(6) the composite $S^{17} \xrightarrow{\overline{\sigma^2}} S^1 \cup_\eta e^3 \xrightarrow{\tilde{\nu}} S^{-2}$ is always $\bar{\sigma}$ and is detected by c_1.

Consider $\Sigma^{-1}P_1^4 = S^0 \cup_{2\iota} e^1 \cup e^2 \cup_{2\iota} e^3$ which is the mapping cone of $S^2 \xrightarrow{(\bar{\eta}, 2\iota)} (S^0 \cup_{2\iota} e^1) \vee S^2$, where $\bar{\eta}$ is a coextension of $\eta\colon S^2 \to S^1$. There is an obvious map

$$\Sigma^{-1}P_1^4 \xrightarrow{q} S^1 \cup_\eta e^3$$

obtained by pinching $S^0 \cup e^2 = S^0 \vee S^2$ to a point. Toda also shows in **[15]** that $\langle 2\iota, \eta, \sigma^2\rangle = \{0\}$ and $2\sigma^2 = 0$. These imply that there is a map $\overline{\overline{\sigma^2}}\colon S^{17} \to \Sigma^{-1}P_1^4$ such that

(7) the composite $S^{17} \xrightarrow{\overline{\overline{\sigma^2}}} \Sigma^{-1}P_1^4 \xrightarrow{q} S^1 \cup_\eta e^3$ is a $\overline{\sigma^2}$.

We will show that there is a map $\bar{g}\colon \Sigma^{-1}P_1^4 \to P^{-2}_{-2^{i-1}-1}$ such that

(8) the composite $\Sigma^{-1}P_1^4 \xrightarrow{\bar{g}} P^{-2}_{-2^{i-1}-1} \xrightarrow{p} S^{-2}$ is equal to the composite $\Sigma^{-1}P_1^4 \xrightarrow{q} S^1 \cup_\eta e^3 \xrightarrow{\tilde{\nu}} S^{-2}$ for some $\tilde{\nu}$.

(6), (7) and (8) imply Proposition 2(b) by taking $g = \bar{g} \circ \overline{\overline{\sigma^2}}$.

To show (8) consider the cofibration sequence

$$\Sigma^{-1}P^{-1}_{-2^{i-1}-1} \to \Sigma^{-1}P^4_{-2^{i-1}-1} \to \Sigma^{-1}P_0^4 \xrightarrow{\tilde{g}} P^{-1}_{-2^{i-1}-1}$$

induced by the inclusion $\Sigma^{-1}P^{-1}_{-2^{i-1}-1} \to \Sigma^{-1}P^4_{-2^{i-1}-1}$. Here $\tilde{g}$ is not uniquely determined; it can be altered, say, by any self-homotopy equivalence $\Sigma^{-1}P_0^4 \to \Sigma^{-1}P_0^4$. I. M. James shows in **[10]** that $P^{-1}_{-2^{i-1}-1} \simeq P^{-2}_{-2^{i-1}-1} \vee S^{-1}$ and $\Sigma^{-1}P_0^4 \simeq S^{-1} \vee \Sigma^{-1}P_1^4$. Consider the composite

$$\bar{g}\colon \Sigma^{-1}P_1^4 \xrightarrow{\tilde{g}|\Sigma^{-1}P_1^4} P^{-1}_{-2^{i-1}-1} \simeq P^{-2}_{-2^{i-1}-1} \vee S^{-1} \to P^{-2}_{-2^{i-1}-1},$$

where the second map is the projection. We will show that there is a choice of $\tilde{g}$ such that

$$\text{(9)}\qquad \text{the composite } \Sigma^{-1}P_1^4 \xrightarrow{\tilde{g}} P_{-2^{j-1}-1}^{-2} \xrightarrow{p} S^{-2} \text{ is zero on the bottom cell of } \Sigma^{-1}P_1^4.$$

Granting (9) we see that, since $\pi_4^s = 0$, $p \circ \tilde{g}$ is trivial on the subcomplex $S^0 \cup e^2 = S^0 \vee S^2$ and therefore factorizes through $S^1 \cup_\eta e^3$, and it is easy to see that the factorization map $S^1 \cup_\eta e^3 \to S^{-2}$ is a $\tilde{\nu}$. Thus $\tilde{g}$ satisfies (8).

Suppose $\lambda = p \circ (\tilde{g}|S^0)\colon S^0 \to S^{-2}$ is nonzero; so $\lambda = \eta^2$. It is easy to see that the composite

$$S^{-1} \xrightarrow{\tilde{g}|S^{-1}} P_{-2^{j-1}-1}^{-1} \simeq P_{-2^{j-1}-1}^{-2} \vee S^{-1} \to P_{-2^{j-1}-1}^{-2} \xrightarrow{p} S^{-2}$$

is η. We have $2\nu \in \langle \eta, 2\iota, \eta \rangle$ **[15]**. Hence there is a map $\eta'\colon \Sigma^{-1}P_1^4 \to S^{-1}$ which, when restricted to S^0, is η. Consider the map

$$\Sigma^{-1}P_0^4 = S^{-1} \vee \Sigma^{-1}P_1^4 \xrightarrow{\alpha = \begin{bmatrix} 1 & \eta' \\ 0 & 1 \end{bmatrix}} \Sigma^{-1}P_0^4 = S^{-1} \vee \Sigma^{-1}P_1^4.$$

It is clear that α is a self-homotopy equivalence. Let $\tilde{g}' = \tilde{g} \circ \alpha\colon \Sigma^{-1}P_0^4 \to P_{-2^{j-1}-1}^{-1}$. Then the corresponding $\bar{g}'$ satisfies (9). This completes the proof.

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