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A p-LOCAL SPLITTING OF $BU(n)$

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ABSTRACT. Let p be a prime and let $n > 1$. A necessary and sufficient condition that the classifying space $BU(n)$ is p-equivalent to the product of nontrivial spaces is that p does not divide n.

Let $U(n)$ denote the Lie group of unitary $n \times n$ matrices, and let $U = \varinjlim U(n)$. In this paper we study the classifying space $BU(n)$ and determine those primes at which this space is equivalent to a product. The result is quite different from the infinite case. Recall that when we pass to the limit there are two types of splitting that occur. The first requires no localization;

$$BU \simeq BT^1 \times BSU.$$

The proof of this splitting is elementary, of course, but it does use the H-structure on BU. The second type of splitting is truly p-primary. At each prime p, BU splits into a product of p irreducible spaces

$$BU \simeq_p \prod_{n=1}^{p} B(2n, p).$$

This was first proved by Peterson [**6**]. A thorough account of this splitting is also given in Zabrodsky's book [**8**].

The main result of this paper is

THEOREM. *If $1 < n < \infty$, then $BU(n)$ is irreducible at p if and only if p divides n. If p does not divide n, then*

$$BU(n) \simeq_p BT^1 \times BSU(n)$$

and both factors are irreducible.

Most of the work in our proof involves showing that when p divides n, the unstable algebra $H^*(BU(n); \mathbf{F}_p)$ is indecomposable over the Steenrod algebra. In other words, it cannot be expressed as the tensor product of two nontrivial unstable A^*-algebras. Here A^* denotes the Steenrod algebra modulo the two-sided ideal generated by the Bockstein coboundary. Our proof uses reflection groups and the methods and results of Adams and Wilkerson [**2**].

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1. A^*-algebras. Let H^* and E^* be A^*-algebras. We say that E^* is a *retract* of H^* if there are A^*-maps

$$E^* \underset{\pi}{\overset{i}{\rightleftarrows}} H^*$$

such that $\pi \cdot i = 1_{E^*}$.

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PROPOSITION 1. *Suppose that* $H^* \cong H^*(BT^n : \mathbf{F}_p)^W$ *where* W *is a suitable group of* A^**-automorphisms. Then any* A^**-retract of* H^* *is likewise the ring of invariants in* $H^*(BT^k : \mathbf{F}_p)$ *for some integer* k *and some group* W'.

PROOF. The argument uses the main result of Adams and Wilkerson [**2**] and the naturality of A^*-maps. Suppose that E^* is a retract of H^*. Obviously, E^* is embedded in $H^*(BT^n : \mathbf{F}_p)$. Since $E^* = \pi H^*$, it is Noetherian. So it remains to show that E^* satisfies the two conditions in [**2**, Theorem 1.2]:

(i) E^* is integrally closed in its field of fractions.

(ii) If $y \in E^{2dp}$ and $Q^r y = 0$ for any $r \geq 1$, then $y = x^p$ for some $x \in E^{2d}$.

First, suppose $\alpha \in q(E^*)$. Here $q(R)$ denotes the quotient field of an integral domain R. Let $\tilde{i}$ be the monomorphism of the fields $q(E^*) \to q(H^*)$ such that $\tilde{i}|_{E^*} = i$. If α is integral over E^*, then the image $\tilde{i}(\alpha)$ is integral over H^*. Since H^* is integrally closed, $\tilde{i}(\alpha)$ lies in H^*. Let us write $\alpha = \alpha_1/\alpha_2$ where $\alpha_i \in E^*$. Thus, we get $i(\alpha_1) = i(\alpha_2) \cdot \beta_0$ for some $\beta_0 \in H^*$. Applying the map π, it follows that

$$\pi \cdot i(\alpha_1) = \pi \cdot i(\alpha_2) \cdot \pi(\beta_0), \qquad \alpha_1 = \alpha_2 \cdot \pi(\beta_0).$$

Since $\pi(\beta_0) \in E^*$, we conclude that α lies in E^*. So E^* is integrally closed. Next, suppose $y \in E^{2dp}$ and $Q^r y = 0$ for any $r \geq 1$. Since i is an A^*-map, then $Q^r i(y) = 0$. Thus there is $x \in H^{2d}$ such that $i(y) = x^p$.

Once again we apply the map π, getting $\pi i(y) = \pi(x^p), y = \pi(x)^p$ where $\pi(x) \in E^{2d}$. This completes the proof.

2. Generalized reflection groups. Let V be a finite-dimensional vector space over a field k. A *pseudo-reflection* of V is a linear automorphism w such that $\operatorname{rank}(1-w) = 1$. We say that a vector u is a *direction* of a pseudo-reflection if it is an eigenvector for the eigenvalue that is not equal to 1.

Let $\rho: G \to \mathrm{GL}(V)$ be a linear representation. A nonzero vector $t \in V$ is called *G-invariant* if $\rho(g)t = t$ for any $g \in G$. The representation ρ is called *reducible* with respect to a G-invariant vector t if there is a hyperplane V_0 in V such that $V = V_0 \oplus \langle t \rangle$ and, for any $g \in G$, the automorphism $\rho(g)$ has the form $\gamma \oplus 1$ for some $\gamma \in \mathrm{GL}(V_0)$.

PROPOSITION 2. *Let* W *be the group generated by pseudo-reflections* $w_1, \ldots, w_r$. *Assume that each* u_i *is a direction of* w_i *and that* t *is* W*-invariant. Then* W *is reducible with respect to* t *if and only if the vector* t *does not belong to the subspace spanned by* $u_1, \ldots, u_r$.

PROOF. Suppose that W is reducible and that V_0 is the hyperplane. Let w be one of the generators $w_1, \ldots, w_r$ and let u be a direction of w. We can write $u = v_0 + bt$ for some $v_0 \in V_0$ and $b \in k$. If a is the eigenvalue which is not 1, it follows that

$$0 = w(u) - au = w(v_0 + bt) - a(v_0 + bt) = w(v_0) - av_0 + b(1-a)t.$$

Since $w(v_0) \in V_0$, we get $b(1-a) = 0$. So $b = 0$ and $u = v_0 \in V_0$. This shows that $\operatorname{Span}(u_1, \ldots, u_r) \subset V_0$. Therefore, t does not belong to $\operatorname{Span}(u_1, \ldots, u_r)$.

Conversely, if $t \notin \operatorname{Span}(u_1, \ldots, u_r)$, then there is a hyperplane V_0 such that $\operatorname{Span}(u_1, \ldots, u_r) \subset V_0$ and $V = V_0 \oplus \langle t \rangle$. Given a generator w with direction u, we have a decomposition; $V = \langle u\rangle \oplus \mathrm{Ker}(w-1)$. Let us write $V_w = V_0 \cap \mathrm{Ker}(w-1)$. We claim $V_0 = \langle u\rangle \oplus V_w$. In fact, we see that $\mathrm{Ker}(w-1) = V_w \oplus U$ for some subspace U. Since $V_0 \cap U = 0$, we get $V_0 \oplus U \subset V$ so that $\dim U \leq 1$ and hence $\dim V_w \geq n-2$. We notice that $V_w \neq \mathrm{Ker}(w-1)$ since $t \notin V_0$. Therefore, $\dim V_w = n-2$. We now see that $w \cdot V_0 \subset V_0$ since $w(\langle u\rangle) \subset \langle u\rangle$ and w fixes V_w pointwise. Thus V_0 is invariant under the W-action and hence W is reducible with respect to t. This completes the proof.

3. Proof of the Theorem. First assume that p divides n. By Borel [**3**, Proposition 29.2], we see that $H^*(BU(n) : \mathbf{F}_p) = H^*(BT^n : \mathbf{F}_p)^{S_n}$ where S_n is the symmetric group. Suppose that $H^*(BU(n) : \mathbf{F}_p)$ is A^*-decomposable. According to Proposition 1, there is an A^*-isomorphism θ from $H^*(BU(n) : \mathbf{F}_p)$ to $H^*(BT^{n_1} : \mathbf{F}_p)^{W_1} \otimes H^*(BT^{n_2} : \mathbf{F}_p)^{W_2}$ for some integers n_1 and n_2 and some suitable groups W_1 and W_2 because each A^*-algebra is a retract. By Adams and Wilkerson [**2**, Proposition 1.10], we can find an A^*-map ϕ which makes the following diagram commutative:

$$\begin{array}{ccc} H^*(BT^n : \mathbf{F}_p) & \xrightarrow{\phi} & H^*(BT^{n_1+n_2} : \mathbf{F}_p) \\ \uparrow & & \uparrow \\ H^*(BU(n) : \mathbf{F}_p) & \underset{\cong}{\xrightarrow{\theta}} & H^*(BT^{n_1} : \mathbf{F}_p)^{W_1} \otimes H^*(BT^{n_2} : \mathbf{F}_p)^{W_2}. \end{array}$$

In this diagram the vertical maps are injective. If $W = W_1 \times W_2$, then clearly

$$H^*(BT^{n_1} : \mathbf{F}_p)^{W_1} \otimes H^*(BT^{n_2} : \mathbf{F}_p)^{W_2} = H^*(BT^{n_1+n_2} : \mathbf{F}_p)^{W}.$$

Recall that $H^*(BU(n) : \mathbf{F}_p)$ is a polynomial ring in n variables. Thus the maximum number of elements in $H^*(BT^{n_1+n_2} : \mathbf{F}_p)^W$ which can be algebraically independent over $\mathbf{F}_p$ is n; so we have $n_1 + n_2 = n$.

Recall that $H^*(BU(n) : \mathbf{F}_p) \hookrightarrow H^*(BT^n : \mathbf{F}_p)$ is a Galois extension with Galois group S_n. Lang [**5**, p. 247] shows that for any $w \in W$ there exists $\sigma \in S_n$ such that $w\phi = \phi\sigma$. We claim that ϕ is invertible. In fact, if an A^*-map ψ covers θ^{-1}, then $\psi \cdot \phi$ covers $\theta^{-1} \cdot \theta =$ identity; so the map $\psi \cdot \phi$ differs from the identity map by a permutation. Thus ϕ is injective and hence bijective for dimensional reason. Consequently $\sigma = \phi^{-1} w \phi$. Thus it follows that, if $H^*(BU(n) : \mathbf{F}_p)$ is A^*-decomposable, then the group S_n is conjugate to $W_1 \times W_2$ in $\mathrm{GL}(n : \mathbf{F}_p)$. It is well known that the symmetric group is not the product of two nontrivial subgroups. Consequently one of the W_i's must be trivial and it follows that this representation of S_n is reducible with respect to an S_n-invariant vector.

Regard $H^2(BT^n : \mathbf{F}_p)$ as a vector space over $\mathbf{F}_p$ with basis $t_1, \ldots, t_n$. The symmetric group acts on this vector space by the rule $\sigma(t_i) = t_{\sigma(i)}$. Recall that S_n is generated by the transpositions $\sigma_1, \ldots, \sigma_{n-1}$ where $\sigma_i = (i, i+1)$ and that the vector $t = \sum_{i=1}^n t_i$ is S_n-invariant.

Suppose p is odd. Each σ_i is a pseudo-reflection and the vector $u_i = t_i - t_{i+1}$ is a direction. Since the representation of S_n is reducible with respect to t, Proposition 2 shows that the S_n-invariant vector t does not belong to $\mathrm{Span}(u_1, \ldots, u_{n-1})$. Thus $\{u_1, \ldots, u_{n-1}, t\}$ must be a basis. Equivalently the following $n \times n$ matrix must be nonsingular:

$$\begin{pmatrix} 1 & & & & 1 \\ -1 & \ddots & & 0 & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \ddots \\ & & & -1 & 1 \end{pmatrix}.$$

Since the determinant of the matrix is n, the prime p does not divide n. This contradicts our assumption.

In the case $p = 2$, it is enough to show that there is no such hyperplane V_0 when n is even. We recall that V has basis $t_1, \dots, t_n$. Suppose that V_0 exists. Since S_n acts on V_0, without loss of generality we may assume that $t_1 + \cdots + t_m$ is contained in V_0 for some $m < n$. If $m = 1$, then we can find $\sigma \in S_n$ such that $t_i = \sigma t_1$. Thus each t_i belongs to V_0. But $\dim V_0 = n - 1$, thus $m > 1$. If $m = 2$, then for each $k = 2, \dots, n$ we can find permutations $\tau_1, \dots, \tau_{k-1}$ such that

$$t_1 + t_k = \sum_{r=1}^{k-1} (t_r + t_{r+1}) = \sum_{r=1}^{k-1} \tau_r(t_1 + t_2).$$

Thus, each $t_1 + t_k \in V_0$ and hence $t = \sum_{k=2}^{n} (t_1 + t_k)$ is contained in V_0 since n is even. This contradicts the assumption $V = V_0 \oplus \langle t \rangle$. Therefore, $2 < m < n$. Then we have, however, that

$$t_m + t_{m+1} = t_1 + \cdots + t_m + \sigma_m(t_1 + \cdots + t_m) \in V_0$$

and therefore $t_1 + t_2 \in V_0$. This is also a contradiction. We now conclude that V_0 does not exist.

Next assume that p does not divide n. Consider the map $f\colon T^1 \times SU(n) \to U(n)$ given by

$$f(z, A) = \begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} \cdot A$$

where $z \in T^1$ and $A \in SU(n)$. It is easy to see that this map is a homomorphism with fibre $\mathbf{Z}/n$. On the level of classifying spaces, this map induces another fibration

$$B\mathbf{Z}/n \to BT^1 \times BSU(n) \xrightarrow{Bf} BU(n).$$

Localization preserves fibrations; consequently, when this fibration is localized at p, the fibre $B\mathbf{Z}/n$ becomes contractible since p does not divide n. Hence the map Bf becomes a homotopy equivalence.

It remains to show that $BSU(n)$ and BT^1 are irreducible at p. For BT^1, this is obvious because $BT^1_{(p)} = K(\mathbf{Z}_{(p)}, 2)$. For $BSU(n)$, the argument is very similar to the one used before. Namely, if $BSU(n)$ split as a product at the prime p, then it would follow that the representation of its Weyl group S_n in $\mathrm{GL}(n-1\colon \mathbf{F}_p)$ would be conjugate to a product. Just as before, it would follow that this representation would, in fact, be reducible with respect to a nonzero S_n-invariant vector t'. But such a vector would correspond to a generator of $H^2(BSU(n) : \mathbf{F}_p) = 0$. This contradiction completes the proof of the Theorem.

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