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ON A QUESTION OF ARCHANGELSKIJ CONCERNING LINDELÖF SPACES WITH COUNTABLE PSEUDOCHARACTER

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ABSTRACT. We give a negative solution to Archangelskij's problem by showing that there exists a Lindelöf space with countable pseudocharacter which does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

The aim of this note is to construct, in the usual axioms of set theory, an example mentioned in the abstract (see [**Ar**, Hypotheses 5.4 and 5.5]). S. Shelah obtained, under some set-theoretical assumptions, a Lindelöf space of cardinality greater than 2^ω with countable pseudocharacter (see [**S** and **HJ**]). From the well-known Archangelskij theorem it follows that Shelah's space does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

Let us denote by Q the set of rational numbers of the unit interval. The symbols ω and ω_1 stand for the first infinite and first uncountable ordinal numbers respectively.

EXAMPLE. *There is a Lindelöf space X with countable pseudocharacter which does not admit a continuous one-to-one mapping onto a Hausdorff space satisfying the first axiom of countability.*

Construction of X. There exists a family $\{A_\alpha: 1 \leqslant \alpha < \omega_1\}$ such that

(1) A_α is a countable set consisting of strictly increasing sequences of Q of length α for $1 \leqslant \alpha < \omega_1$;

(2) if $\alpha < \beta < \omega_1$, then $p_\alpha(A_\beta) = A_\alpha$; p_α stands for the projection onto the first α coordinates;

(3) if $a \in A_\alpha$ for $1 \leqslant \alpha < \omega_1$, then for every limit ordinal number $\beta < \alpha$, $a(\beta) = \sup\{a(\lambda): \lambda < \beta\}$ and $\sup\{a(\lambda): \lambda < \alpha\}$ are rational numbers of Q;

(4) if $\alpha < \beta < \omega_1$, $a \in A_{\alpha+1}$, $r \in Q$ and $a(\alpha) < r$, then there exists $b \in A_{\beta+1}$ such that $p_{\alpha+1}(b) = a$ and $b(\beta) = r$ (see [**J**, p. 91, the construction of the Aronszajn tree]).

Let us attach to $a \in A_\alpha$, for $1 \leqslant \alpha < \omega_1$, $x_a \in Q^{\omega_1}$ defined by

$$x_a(\beta) = \begin{cases} a(\beta), & \text{if } \beta < \alpha, \\ \sup\{a(\lambda): \lambda < \alpha\}, & \text{if } \beta \geqslant \alpha. \end{cases}$$

Let $X = \bigcup\{X_\alpha: 1 \leqslant \alpha < \omega_1\}$, where $X_\alpha = \{x_a: a \in A_\alpha\}$, be a subspace of Q^{ω_1}. In [**Al**] it was proved that $Y \times X^\omega$ is Lindelöf provided that Y is a hereditarily Lindelöf space. We shall sketch the proof of the Lindelöf property in X for the sake of

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completeness. Let $\mathscr{B}$ be a countable base of Q and $\mathscr{V}$ an open covering of X. For $x \in X_\alpha$, $\alpha < \omega_1$, let

$$\mathscr{A}(x) = \Big\{B \in \mathscr{B}: x(\alpha) \in B \text{ and there are } \alpha < \beta(x, B) < \omega_1 \text{ and } V \in \mathscr{V} \text{ such that } F(x, B, \beta(x, B)) = \Big(\prod_{\lambda < \omega_1} F_\lambda\Big) \cap X \subset V\Big\},$$

where

$$F_\lambda = \begin{cases} \{x(\lambda)\}, & \text{if } \lambda \leqslant \alpha, \\ B, & \text{if } \lambda = \mathscr{B}(x, B), \\ Q, & \text{otherwise.} \end{cases}$$

Since X consists of increasing sequences, $\mathscr{A}(x) \neq \varnothing$ for every $x \in X$. Let

$$\beta_1 = \sup\{\beta(x, B): x \in X_1 \text{ and } B \in \mathscr{A}(x)\}.$$

Since X_1 and $\mathscr{A}(x)$, for $x \in X_1$, are countable sets, $\beta_1 < \omega_1$. If β_n is defined, then let

$$\beta_{n+1} = \sup\{\beta(x, B): x \in \bigcup\{X_\lambda: \lambda \leqslant \beta_n + 1\} \text{ and } B \in \mathscr{A}(x)\}$$

and

$$\beta = \sup\{\beta_n: n \in N\}.$$

To finish the proof of the Lindelöf property of X it is enough to show that

$$X = \bigcup\{F(x, B, \beta(x, B)): x \in \bigcup\{X_\lambda: \lambda < \beta\} \text{ and } B \in \mathscr{A}(x)\}.$$

Let x be an element of X_λ for $\lambda \geqslant \beta$. Then $p_\beta(x)$ belongs to A_β. Let x' be a point of X_β such that $p_\beta(x') = p_\beta(x)$. There exist $B \in \mathscr{B}$, α_1 and α_2 and $V \in \mathscr{V}$ such that $x'(\beta) \in B$, $\alpha_1 < \beta < \alpha_2$ and $F = (\prod_{\lambda < \omega_1} F_\lambda) \cap X \subset V$, where

$$F_\lambda = \begin{cases} \{x'(\lambda)\}, & \text{if } \lambda < \alpha_1, \\ B, & \text{if } \lambda = \alpha_2, \\ Q, & \text{otherwise.} \end{cases}$$

Without loss of generality we can assume that $\sup\{x'(\lambda): \lambda < \alpha_1\} \in B$. Let v be an element of X_{α_1} such that $p_{\alpha_1}(v) = p_{\alpha_1}(x')$. Then $B \in \mathscr{A}(v)$ and $\beta(v, B) < \beta < \alpha_2$. It is easy to see that $x' \in F(v, B, \beta(v, B))$. Since $p_\beta^{-1} p_\beta(F(v, B, \beta(v, B))) = F(v, B, \beta(v, B))$ and $p_\beta(x') = p_\beta(x)$, $x \in F(v, B, \beta(v, B))$. We conclude that X is a Lindelöf space.

If $x \in X_\alpha$, $\alpha < \omega_1$, then $\{x' \in X: p_{\alpha+2}(x') = p_{\alpha+2}(x)\} = \{x\}$ and $p_{\alpha+2}(X)$ is countable, so $\{x\}$ is a G_δ-subset of X.

To finish the proof of the properties of X it is enough to show that X does not admit a weaker Hausdorff topology τ which satisfies the first axiom of countability. Suppose not and let τ be a weaker Hausdorff topology on X satisfying the first axiom of countability. If $x \in X$ then there exists $\beta(x) < \omega_1$ such that for every open, in τ, neighbourhood U of x there is a basic open neighbourhood $B(U) = (\prod_{\lambda < \omega_1} B_\lambda(U)) \cap X$ of x, with respect to the Tychonoff topology, such that $B_\lambda(U) = Q$ for $\lambda \geqslant \beta(x)$ and $B(U) \subset U$. The existence of $\beta(x)$ is an immediate consequence of the fact that τ satisfies the first axiom of countability. Let $\beta_1 = \sup\{\beta(x): x \in X_1\}$. If β_n is defined, then let

$$\beta_{n+1} = \sup\{\beta(x): x \in \bigcup\{X_\lambda: \lambda \leqslant \beta_n + 1\}\}$$

and $\beta = \sup\{\beta_n: n \in N\}$. Notice that β is a limit ordinal number less than ω_1. Let x_1 and x_2 be points of X_β and $X_{\beta+2}$, respectively, such that $p_{\beta+1}(x_1) = p_{\beta+1}(x_2)$ and $x_1(\beta+1) < x_2(\beta+1)$. To prove that τ is not a Hausdorff topology it is enough to show that if U is an open neighbourhood of x_1, with respect to τ, then there is a sequence $(y_n)_{n=1}^{\infty}$ of points of U converging to x_2, with respect to the Tychonoff topology. Let $B \in \mathscr{B}$ and $\alpha_1, \alpha_2 < \omega_1$ be such that $\beta_1 < \alpha_1 < \beta < \alpha_2$, $x_1(\lambda) \in B$ if $\lambda \geqslant \alpha_1$ and $F = (\prod_{\lambda<\omega_1} F_\lambda) \cap X \subset U$, where

$$F_\lambda = \begin{cases} \{x_1(\lambda)\}, & \text{if } \lambda \leqslant \alpha_1, \\ B, & \text{if } \lambda = \alpha_2, \\ Q, & \text{otherwise.} \end{cases}$$

Let z_n be a point of X_{α_n+1}, where $\alpha_n = \max\{\beta_n, \alpha_1\}$, such that $p_{\alpha_n+1}(z_n) = p_{\alpha_n+1}(x_1)$. Since β is a limit ordinal number, $\alpha_n + 1 < \beta$. By the definition of $\beta(z_n)$, there exists a basic open neighbourhood $G(z_n) = (\prod_{\lambda<\omega_1} G_\lambda(z_n)) \cap X$ of z_n, with respect to Tychonoff topology, such that $G_\lambda(z_n) = Q$, if $\lambda \geqslant \beta(z_n)$ and $G(z_n) \subset U$. Notice that $\beta(z_n) < \beta$ for $n \in N$. Let y_n be an element of $G(z_n) \cap X_{\beta+2}$ such that $p_{\alpha_n+1}(z_n) = p_{\alpha_n+1}(y_n)$ and $y_n(\lambda) = x_2(\lambda)$ for $\lambda \geqslant \beta$. The existence of y_n is an easy consequence of (4), $\beta(z_n) < \beta$, $z_n(\lambda) \leqslant x_1(\lambda)$, for $\lambda < \omega_1$, $p_{\beta+1}(x_1) = p_{\beta+1}(x_2)$, and $x_1(\beta+1) < x_2(\beta+1)$. If $G = (\prod_{\lambda<\omega_1} G_\lambda) \cap X$ is a basic open neighbourhood of x_2, in the Tychonoff topology, $\alpha = \sup\{\lambda < \beta: G_\lambda \neq Q\}$ and k is such that $\beta_k > \alpha$, then $y_n \in G$ provided that $n \geqslant k$, so we conclude that $(y_n)_{n=1}^{\infty}$ converges to x_2 in the Tychonoff topology.

REMARK. Let Z be a subspace of I^{ω_1}, where I stands for the unit interval, of all points of I satisfying the following conditions:

(i) for every $\varepsilon > 0$ and $z \in Z$, $\{\alpha < \omega_1: z(\alpha) > \varepsilon\}$ is finite;

(ii) for every $z \in Z$, $\{\alpha < \omega_1: z(\alpha) > 0\}$ is an initial interval of ω_1.

It is easy to see that Z has countable pseudocharacter. In [C] it was proved that Z has the Lindelöf property. Using our method one can show that Z does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

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