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BOUNDS ON THE MAXIMUM NUMBER OF VECTORS WITH GIVEN SCALAR PRODUCTS

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ABSTRACT. Suppose M, L are sets of real numbers, $V = \{v_1, \dots, v_m\}$ is a collection of vectors in R^n, having k nonzero coordinates all from M and satisfying $(v_i, v_j) \in L$ for $i \neq j$. Theorem 1.1 establishes a polynomial upper bound for $|V|$, generalizing previous results for subsets of a set and $(0, \pm 1)$-vectors. Theorem 1.4 asserts that if $|L| = s$ then $|V| \leqslant \binom{n+s}{s}$. For $M = \{-1, 1\}$, $L = [-(k-1), k-1]$. Theorem 1.5 gives $|V| \leqslant 2^{k-1}\binom{n}{k-1}/k$, where equality holds if and only if V is a "signed" $(n, k, k-1)$ Steiner-system.

1. Introduction. Let $V = \{v_1, \dots, v_m\}$ be a set of vectors in R^n, the n-dimensional Euclidean space. For L, M subsets of real numbers, $0 \notin M$ and a positive integer k, we define $\mathscr{V}(n, k, M, L)$ as the collection of all V satisfying the following three conditions:

(i) v_i has exactly k nonzero coordinates,

(ii) each nonzero coordinate of v_i is from M,

(iii) $(v_i, v_j) \in L$ holds for $1 \leqslant i < j \leqslant m$.

The aim of this paper is to present upper bounds for the maximum size of V with $V \in \mathscr{V}(n, k, M, L)$.

Let us denote this maximum by $m(n, k, M, L)$. Note that $m(n, k, M, L)$ can be infinite for some choices of the parameters. For $M = \{1\}$ this problem has been widely investigated (cf. e.g. **[FrW]**). The case $M = \{-1, +1\}$ was the subject of **[DF1]**.

Define $M^+ = \{m \in M: m > 0\}$, $M^- = \{-m: m \in M, m < 0\}$.

Let us also define $l_{\sup} = \sup\{l: l \in L\}$, $m_{\inf} = \inf\{|m|: m \in M\}$.

THEOREM 1.1. *Suppose that L^+ can be covered by r intervals, each of length less than $m_{\inf}^2/2$; then there exists a constant $c(k, l_{\sup}, m_{\inf})$ depending only on k, $l_{\sup}$ and $m_{\inf}$ such that we have*

$$m(n, k, M, L) \leqslant c(k, l_{\sup}, m_{\inf})\binom{n}{r} m(k, k, M^+ \cup M^-, L^+).$$

The determination of $m(k, k, M, L)$ is a number theoretic question. It is easy to give examples for $m(k, k, M, L) = \infty$ (e.g. $L = Q = \{\text{rationals}\}$, $|M \cap Q| = \infty$) or for $m(k, k, M, L) = 1$ (e.g. $L = \{\text{irrationals}\}$, $M \subset Q$). Note that for $|M|$ finite

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$m(k, k, M, L) \leqslant |M|^k$ holds. It is a consequence of Theorem 1 in **[Fr]** that $m(k, k, M, \{1\}) < (2k)^{2k+1}$. We prove

PROPOSITION 1.2. *If $|L^+|$ is finite then*

$$m(k, k, M, L) \leqslant (2e)^k \binom{|L^+|+k}{k}.$$

After preparations made in §2 we prove Theorem 1.1 in §3.

Let us note the following

COROLLARY 1.3. *If $|L^+| \leqslant r$ and $m_{\inf} > 0$, then we have $m(n, k, M, L) \leqslant c(k, L, M)\binom{n}{r}$.*

Note that for $M = \{1\}$ the statement of the corollary (with $c(k, L, M) = 1$) is a classical theorem of Ray-Chaudhuri and Wilson **[RW]**.

For $|L|$ fixed we have the following more general bound:

THEOREM 1.4. *Suppose $V = \{v_1, \ldots, v_m\}$ is a set of distinct vectors in R^n so that, for each $v \in V$ and for a fixed positive integer s, (v, v') takes up at most s values (i.e., $|\{(v, v'): v \neq v' \in V\}| \leqslant s$). Then $|V| \leqslant \binom{n+s}{s}$.*

Note the analogy with the following result of Bannai, Bannai, Stanton and Blokhuis. Recall that $S \subset R^n$ is called an s-distance set if $|\{d(x, y): x, y \in S, x \neq y\}| = s$ ($d(x, y)$ is the euclidean distance of x and y).

THEOREM **[BBS, B1]**. *If S is an s-distance set in R^n, then $|S| \leqslant \binom{n+s}{s}$.*

Let us note also that if all vectors have equal length, say b, then $d(x, y)^2 = (x - y, x - y) = 2b - 2(x, y)$. That is, the number of different distances and scalar products is the same. For this case there exists the stronger bound of Delsarte, Goethals and Seidel:

THEOREM **[DGS]**. *If S is an s-distance set on the unit sphere in R^n, then $|S| \leqslant \binom{n+s-1}{s} + \binom{n+s-2}{s-1}$.*

Recall that a (n, k, t) partial Steiner-system ζ is a family of k-subsets of $\{1, 2, \ldots, n\}$ such that every t-element subset is contained in at most one member of ζ. Clearly ζ satisfies $|\zeta| \leqslant \binom{n}{t}/\binom{k}{t}$ ($n \geqslant k > t \geqslant 1$). In case of equality, ζ is called a Steiner-system. Very few Steiner-systems with $t > 3$ and no Steiner-systems with $t \geqslant 6$ are known. However, recently Rödl **[Rö]** showed that for fixed k, t and n tending to infinity there exist partial Steiner-systems satisfying

$$|\zeta| \geqslant (1 - o(1))\binom{n}{t}\Big/\binom{k}{t}.$$

Given a partial Steiner-system $\mathscr{P}$ we may replace each $P \in \mathscr{P}$ by a collection of vectors $V_P \in \mathscr{V}(k, k, \{\pm 1\}, \{0, \pm 1, \ldots, \pm(t-1)\})$. Then $V_{\mathscr{P}} = \bigcup_{P \in \mathscr{P}} V_P$: $P \in \mathscr{P}$ is in $\mathscr{V}(n, k, \{\pm 1\}, [-(t-1), t-1])$.

The maximum possible size of V_P is a coding theoretical problem. Here we consider only the case $t = k - 1$. Then the only restriction on $v, v' \in V_P$ is that $v \neq -v'$. Consequently, $|V_P| \leqslant 2^{k-1}$ and there are $2^{2^{k-1}}$ different ways to achieve equality here, namely choose exactly one out of each pair $\{v, -v\}$. If $|V_P| = 2^{k-1}$ for all $P \in \mathscr{P}$, then $V_{\mathscr{P}}$ is called an *antipodally signed* partial Steiner-system.

THEOREM 1.5. *Suppose* $V \in \mathscr{V}(n, k, \{\pm 1\}, [-(t-1), t-1])$. *Then*

$$|V| \leqslant \frac{\binom{n}{t}}{\binom{k}{t}} 2^{t-1}(k - t + 1),$$

moreover, for $k = t + 1$, *equality is possible if and only if* $V = V_{\zeta}$ *for some* $(n, k, k - 1)$ *Steiner-system* ζ.

2. General reductions. For a vector $v = (v^1, v^2, \dots, v^n)$ let us define the support $S(v) = \{i: v^i \neq 0, 1 \leqslant i \leqslant n\}$.

LEMMA 2.1. *For any set of vectors* V *with* $|S(v)| = k$ *for all* $v \in V$ *there exists a partition* $X_1 \cup X_2 \cup \cdots \cup X_k = \{1, 2, \dots, n\}$ *such that the set* $V' = \{v \in V: |S(v) \cap X_i| = 1, i = 1, \dots, k\}$ *satisfies* $|V'| \geqslant (k!/k^k)|V|$.

PROOF. Let $X_1 \cup \cdots \cup X_k$ be a random equipartition of $\{1, 2, \dots, n\}$. Thus $|X_i| = \lfloor n/k \rfloor$ or $\lceil n/k \rceil$. For a given $v \in V$ the probability of $|S(v) \cap X_i| = 1$ for $i = 1, \dots, k$ is given by $|X_1| \cdots |X_k|/\binom{n}{k} \geqslant k!/k^k$. Consequently, the expected number of members of V' is at least $(k!/k^k)|V|$ and the statement follows. □

Note that $k!/k^k > e^{-k}$, thus by considering V', we only lose a constant factor. The family V' is called a *transversal* family of vectors.

LEMMA 2.2. *To any transversal family* $V' \in \mathscr{V}(n, k, M, L)$ *there exists* $V'' \in \mathscr{V}(n, k, M^+ \cup M^-, L^+)$ *satisfying* $|V''| \geqslant 2^{-k}|V'|$.

PROOF. Associate with each $v \in V'$ a (± 1)-vector of length k by defining the ith entry $+1$ if and only if the coordinate of v in $S(v) \cap X_i$ is positive. There are 2^k different (± 1)-vectors of length k. Thus we may choose one, say $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, which has been associated with at least $2^{-k}|V'|$ vectors. Let V^* be this collection and let V'' be the collection which we obtain from V^* by applying the orthogonal transformation of multiplying the ith coordinate of $v \in V^*$ by ε_i, $V'' \in \mathscr{V}(n, k, M^+ \cup M^-, L^+)$ follows. □

Note that Lemmas 2.1 and 2.2 imply

COROLLARY 2.3.

$$m(n, k, m, L) \leqslant (2e)^k m(n, k, M^+ \cup M^-, L^+).$$

COROLLARY 2.4.

$$m(n, k, L) \leqslant m(n, k, \{\pm 1\}, L) \leqslant (2e)^k m(n, k, L^+).$$

For a set of vectors V we define $\zeta(V) = \{S(v): v \in V\}$.

PROPOSITION 2.5.

$$|V| \leqslant |\zeta(V)| m(k, k, M, L). \quad \square \tag{2.1}$$

Suppose V is a transversal family of vectors (for the partition $X_1 \cup \cdots \cup X_k$). Then $\zeta(V)$ is a family of transversals. For a subset $H \subset X_1 \cup \cdots \cup X_k$ we define the projection $\pi(H)$ of H by $\pi(H) = \{i: H \cap X_i \neq \varnothing\}$.

Recall that a *sunflower* of size t and with *center* C is a collection $F_1, \ldots, F_t$ of sets with $F_i \cap F_j = C$ for $1 \leqslant i < j \leqslant t$.

A family of subsets $\mathscr{L}$ is called a *meet semilattice* if $L, L' \in \mathscr{L}$ imply $L \cap L' \in \mathscr{L}$.

Also $\mathscr{I}(S, \zeta) = \{S \cap S': S' \in \zeta\}$.

LEMMA 2.6 (FÜREDI [Fü]). *Given k, t, there exists a positive constant $c = c(k, t)$ such that any transversal family ζ of k-subsets contains a subfamily ζ^* satisfying*:

(i) $|\zeta^*| > c(k, t)|\zeta|$,

(ii) *every $C \in \mathscr{I}(S, \zeta^*)$ is the center of a sunflower of size t ($S \in \zeta^*$)*,

(iii) *$\mathscr{I}(S, \zeta^*)$ is a meet semilattice*,

(iv) *the family $\pi(\mathscr{I}(S, \zeta^*)) = \{\pi(C): C \in \mathscr{I}(S, \zeta^*)\}$ is independent of the choice of $S \in \zeta^*$.*

For $\zeta = \zeta(V)$ let us apply Lemma 2.6 with $t = 2$ and set $\mathscr{I}_0 = \pi(\mathscr{I}(S, \zeta^*))$.

A set $G \subset \{1, 2, \ldots, k\}$ is called a *generator set* of $\mathscr{I}_0$ if $G \subset \mathscr{K} \in \mathscr{I}_0$ implies $\mathscr{K} = \{1, 2, \ldots, k\}$.

PROPOSITION 2.7. *Suppose $\mathscr{I}_0$ has a generator set G of size r. Then $|\zeta^*| \leqslant \binom{n}{r}$.*

PROOF. For each $S \in \zeta^*$ choose $G(S) \subset S$ so that $\pi(G(S)) = G$ holds. Now $S' \in \zeta^*$, $G(S) \subset S'$ imply $G(S) \subseteq S \cap S' \in \mathscr{I}(S, \zeta^*)$. Thus $S' = S$ follows. Consequently, the $G(S)$ are all distinct r-subsets of $X_1 \cup \cdots \cup X_k$. This yields $|\zeta^*| \leqslant \binom{n}{r}$. $\square$

Recall that a *chain of length r* is a family $A_0, A_1, \ldots, A_r$ with $A_0 \subset A_1 \subset \cdots \subset A_r$.

PROPOSITION 2.8. *Suppose $\mathscr{I}_0$ contains no chain of length $r + 1$. Then $\mathscr{I}_0$ has a generator set of size at most r.*

PROOF. Apply induction on r. If $r = 1$ then $\mathscr{I}_0$ contains at most one proper subset $\mathscr{B}$ of $\{1, 2, \ldots, k\}$. Thus $\{i\}$ is a generator for all $1 \leqslant i \leqslant k$, $i \notin \mathscr{B}$.

Suppose $\mathscr{B} \subsetneqq \{1, 2, \ldots, k\}$, $\mathscr{B}$ is a maximal element in $\mathscr{I}_0$. Define $\tilde{\mathscr{I}} = \{\mathscr{B} \cap \mathscr{B}_0: \mathscr{B}_0 \in \mathscr{I}_0\}$. Then $\tilde{\mathscr{I}}$ has no chain of length r. Consequently, it has a generator set H of size at most $r - 1$. Now $G = H \cup \{i\}$ is a generator for $\mathscr{I}_0$ whenever $i \in (\{1, 2, \ldots, k\} - H)$. $\square$

3. The proof of Theorem 1.1. In view of Corollary 2.3 and Lemma 2.1 we may assume $V \in \mathscr{V}(n, k, M^+ \cup M^-, L^+)$, V is a transversal family of vectors. By omitting at most n vectors we may suppose no $v \in V$ has a coordinate position $v^{(i)}$ with $v^{(i)} > \sqrt{l_{\sup}}$.

Let us set $\gamma = m_{\inf}^2 / 2k\sqrt{l_{\sup}}$. With $v \in V$, having positive coordinate α_i in X_i we associate a sequence of integers $(t_1, \ldots, t_k)$ where $t_i = \lceil \alpha_i / \gamma \rceil$.

This defines a partition of V into at most $\left[\sqrt{l_{\sup}}/\gamma\right]^k$ parts. This constant factor can be incorporated into $c(k, l_{\sup}, m_{\inf})$, thus we may suppose: the same sequence $(t_1, \ldots, t_k)$ is associated with each $v \in V$.

In view of (2.1) and Lemma 2.6, it will be sufficient to show that $\mathscr{I}_0$—associated with $\zeta^*(V)$—contains no chain of length $r+1$.

Fix $v \in V$ with $S = S(v) \in \zeta^*(V)$.

Suppose for contradiction, $A_0 \subset A_1 \subset \cdots \subset A_r \subset A_{r+1} = \{1, 2, \ldots, k\}$ is a chain in $\mathscr{I}_0$. Let $B_0 \subset B_1 \subset \cdots \subset B_{r+1}$ be the corresponding chain in $\mathscr{I}(S, \zeta^*(V))$. Then for $i = 0, 1, \ldots, r$ there exists $v_i \in V$ with $s(v_i) \cap S = B_i$. By Dirichlet's principle there are two vectors, say v_i, v_j, $i < j$, such that (v, v_i) and (v, v_j) are in the same interval of length less than $m_{\inf}^2/2$.

Taking into consideration $|\beta_\gamma - \beta'_\gamma| < \gamma$ and $(v, v_j) - (v, v_i) < m_{\inf}^2/2$, we infer

$$\frac{1}{2} m_{\inf}^2 > (v, v_j) - (v, v_i) = \sum_{\nu \in B_i} \alpha_\nu (\beta'_\nu - \beta_\nu) + \sum_{\nu \in B_j - B_i} \alpha_\nu \beta'_\nu$$

$$> -\gamma k \sqrt{l_{\sup}} + m_{\inf}^2 \geqslant \frac{1}{2} m_{\inf}^2.$$

A contradiction. □

4. The proof of Proposition 1.2. In view of Corollary 2.3, it is sufficient to consider the case $L = L^+$, and show $m(k, k, M, L) \leqslant \binom{|L|+k}{k}$. However this is a special case of Theorem 1.4. □

5. Exterior products and covering points by hyperplanes. Denote by P^n the real projective space of dimension n. A hyperplane is a subspace of dimension $n-1$.

THEOREM 5.1. *Suppose $T \subset P^n$, $T = \{x_1, \ldots, x_m\}$, s a positive integer and for every point x_i there exist s hyperplanes $H_1^{(i)}, \ldots, H_s^{(i)}$ so that $\{x_i, x_{i+1}, \ldots, x_m\} \cap (H_1^{(i)} \cup \cdots \cup H_s^{(i)}) = \{x_{i+1}, \ldots, x_m\}$. Then $|T| \leqslant \binom{n+s}{s}$.*

PROOF. Let π be a polarity of P^n. Then $\pi(T) = \{\pi(x_i): 1 \leqslant i \leqslant m\}$ is a collection of hyperplanes satisfying the following:

(i) For $1 \leqslant i \leqslant m$ there exist s points $(\pi(H^{(i)}), \ldots, \pi(H_s^{(i)}))$, none of which is on $\pi(x_i)$ so that $\pi(x_j)$ contains at least one of them for $j > i$.

If we replace in (i) the condition $j > i$ by $i \neq j$ then the upper bound $|T| = |\pi(T)| \leqslant \binom{n+s}{s}$ is just Theorem 4.8 in **[Lo]**. A slight modification of Lovász's argument shows that the s-element sets $\{\{\pi(H_1^{(i)}), \ldots, \pi(H_s^{(i)})\}: 1 \leqslant i \leqslant m\}$ which are points in the sth symmetric power of P^n, are independent for $1 \leqslant i \leqslant m$. Consequently, their number m does not exceed the dimension $\binom{n+s}{s}$. □

Next we derive Theorem 1.4.

Arrange the vectors in V so that $|v_1| \geqslant |v_2| \geqslant \cdots \geqslant |v_m|$. Then $(v_i, v_i) > (v_i, v_j)$ holds for $i < j \leqslant m$. Let $\lambda_1^{(i)}, \ldots, \lambda_s^{(i)} < (v_i, v_i)$ be such that $(v_i, v_j) \in \{\lambda_1^{(i)}, \ldots, \lambda_s^{(i)}\}$ holds for all $i < j \leqslant m$. Let us define

$$G_t^{(i)} = \{v \in R^n: (v_1, v) = \lambda_t^{(i)}\}, \qquad 1 \leqslant t \leqslant s.$$

Then $G_t^{(i)}$ is a hyperplane in R^n. Define $H_t^{(i)}$ as the unique hyperplane in P^n containing $G_t^{(i)}$.

Then $\{v_1, \dots, v_n\}$ and the hyperplanes $H_t^{(i)}$ fulfill the assumptions of Theorem 5.1. Consequently, $|V| = m \leqslant \binom{n+s}{s}$. □

As pointed out by the referee, Theorem 1.4 can be deduced also using the approach of Koornwinder [**Ko**].

6. Signed Steiner systems. For a set of vectors $V = \{v_1, \dots, v_m\}$ define $-V = \{-v_1, \dots, -v_m\}$.

PROPOSITION 6.1. *If $L^+ = L^-$ then the following two conditions are equivalent*:
(i) $V \in \mathscr{V}(n, k, \{\pm 1\}, L)$,
(ii) $V \cup (-V) \in \mathscr{V}(n, k, \{\pm 1\}, L \cup \{-k\})$. □

COROLLARY 6.2. *If $-k \notin L$, $L^+ = L^-$ then*

$$2m(n, k, \{\pm 1\}, L) = m(n, k, \{\pm 1\}, L \cup \{-k\})$$

holds. □

In view of Proposition 6.1 and Corollary 6.2, Theorem 1.5 will follow once we proved the next theorem.

THEOREM 6.3. *Suppose* $V \in \mathscr{V}(n, k, \{\pm 1\}, \{0, \pm 1, \dots, \pm(t-1)\} \cup \{-k\})$. *Then*

$$|V| \leqslant \frac{\binom{n}{t}}{\binom{k}{t}} 2^t (k - t + 1). \tag{6.1}$$

Moreover, for $k = t + 1$ equality is possible if and only if there is an (n, k, t) Steiner-system ζ so that V consists of all $(0, \pm 1)$-vectors v with $S(v) \in \zeta$.

PROOF OF THEOREM 6.3. There are $2^t\binom{n}{t}$ $(0, \pm 1)$-vectors with t nonzero coordinates. For such a vector w and $v \in \mathscr{V}$ define $w < v$ if $(w, v) = t$ (or equivalently if $v - w$ is a $(0, \pm 1)$-vector with $k - t$ nonzero entries). Define $V_w = \{v - w : w < v \in V\}$. Then $V_w \in \mathscr{V}(n - t, k - t, \{\pm 1\}, \{-1, -2, \dots, -2t + 1\})$.

A theorem of Delsarte, Goethals and Seidel [**DGS**] yields $|V_w| \leqslant k - t + 1$.

Since for each $v \in V$ there are $\binom{k}{t}$ choices of w, $w < v$, (6.1) follows.

Suppose now equality holds in (6.1), $k = t + 1$. By our argument, $|V_w| = 2$ must hold for all w. Suppose $w < v$, $w < v'$. Since $(v, v') \leqslant k - 2 < t$, we infer $S(v) = S(v')$, and v and v' have opposite sign in $S(v) - S(w)$. Set $S = S(v)$.

We claim that all $(0, \pm 1)$-vectors u with $S(u) = S$ are in V. Indeed, the contrary implies the existence of two $(0, \pm 1)$-vectors u_1, u_2 with $S(u_1) = S(u_2) = S$, $u_1 \in V$, $u_2 \notin V$ and u_1 and u_2 differ in only one position (i.e. $(u_1, u_2) = k - 2$)). Let w_1 be the unique $(0, \pm 1)$-vector with $t = k - 1$ nonzero entries satisfying $w_1 < u_1$, $w_1 < u_2$. The equality in (6.1), as we have shown above, implies, via $|V_{w_1}| = 2$, that $u_2 \in V$, a contradiction.

Now consider $\zeta = \{S(v): v \in V\}$. By the above we have $|V| = 2^k|\zeta|$, that is $|\zeta| = \binom{n}{k-1}/\binom{k}{k-1}$. Clearly $|S \cap S'| \leqslant k - 2$ holds for $S, S' \in \zeta$, thus ζ is a $(n, k, k - 1)$ Steiner-system and the statement follows. □

7. Equidistant sets. In the case $M = \{\pm 1\}$, $L = \{l\}$ one can get tight bounds. For vectors w, v define $w \leqslant v$ if w and v coincide in each nonzero coordinate of w.

THEOREM **[DF2]**. *Suppose* $k > l \geqslant 1$, $V \in \mathscr{V}(n, k, \{\pm 1\}, \{l\})$ *and* $|V| > \max\{(k - l)^2 + (k - l) + 1, (k - l)(l + 2)\}$. *Then there exists a* $(0, \pm 1)$*-vector* w *with* l *nonzero positions so that* $w < v$ *holds for all* $v \in V$ (*and consequently the vectors* $\{v - w: v \in V\}$ *are pairwise orthogonal*).

The corresponding theorem for $M = \{1\}$ was proved in **[De]**.

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