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MONIC POLYNOMIALS AND GENERATING IDEALS EFFICIENTLY

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ABSTRACT. If I is an ideal containing a monic polynomial in $R[T]$ where R is a semilocal ring, then I and I/I^2 require the same minimal number of generators. An ideal containing a monic polynomial in a polynomial ring need not possess any minimal set of generators having a monic as a part of it.

1. Introduction. We are concerned with rings which are commutative and Noetherian with identity. By the dimension of a ring we mean the Krull dimension and we shall have occasion only to deal with rings of finite dimension. Let A be a ring and let M be a finitely generated A-module. We define $\mu(M)$ to be the least number of elements in M required to generate M as an A-module. The conormal bundle of an ideal I in a ring A is the group I/I^2 viewed as an A/I-module. Many algebraic properties of this module are intertwined with those of the ideal I. For instance, the content of an easily verifiable result is that $\mu(I/I^2) \leqslant \mu(I) \leqslant \mu(I/I^2) + 1$. To see when the lower inequality becomes an equality has been the theme of many papers in the literature; and this equality depends heavily on how the ideal I sits inside A. In this article we embark on the following problem: Let $A = R[T]$ be a polynomial ring. Let I be an ideal in A such that I contains a monic polynomial. Is it true that $\mu(I) = \mu(I/I^2)$?

In a lovely paper [4] Satyagopal Mandal has shown that if $\mu(I/I^2) \geqslant \dim(A/I) + 2$ and I contains a monic polynomial then indeed $\mu(I) = \mu(I/I^2)$. While it is not difficult to obtain a positive answer to the question above in the case when R is semilocal, we suspect of no situation where the desired equality between $\mu(I)$ and $\mu(I/I^2)$ will fail. We cite examples of ideals without monic polynomials for which the equality does not hold. One may be curious to know whether a monic polynomial should appear as a part of some minimal set of generators for I given that I does contain monic polynomials. In general, the answer turns out to be in the negative.

2. Cases when equality holds. Let us begin with a simple but useful lemma.

LEMMA 2.1. *If an ideal I in a ring A is contained in all but finitely many maximal ideals of A, then $\mu(I) = \mu(I/I^2)$.*

PROOF. Let $\mu(I/I^2) = n$. Choose elements $a_1, \ldots, a_n$ in I which generate I mod I^2. If I is contained in all the maximal ideals of A, then $(a_1, \ldots, a_n) = I$. Otherwise, let

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$M_1, \ldots, M_r$ be all those maximal ideals of A that do not contain I. We can rearrange these M_i's to assume that a_1 is in $M_1, \ldots, M_t$ but not in $M_{t+1}, \ldots, M_r$. Then the ideal $J = I^2 \cap M_{t+1} \cap \cdots \cap M_r$ is not contained in $M_1 \cup M_2 \cup \cdots \cup M_t$. We can find an element b in J such that b is outside $M_1 \cup M_2 \cup \cdots \cup M_t$. Replace a_1 by $a_1 + b = a$. Then $a, a_2, \ldots, a_n$ generate I since they do so locally.

COROLLARY 2.2. *Let A be a semilocal ring. Then $\mu(I) = \mu(I/I^2)$ for any ideal I in A.*

As a consequence we obtain

THEOREM 2.3. *Let $A = R[X]$ where R is a semilocal ring. Let I be an ideal in A such that I contains a monic polynomial. Then $\mu(I) = \mu(I/I^2)$. Further, we can find a minimal set of generators for I such that one (hence all) elements in this set are monic polynomials.*

PROOF. Let $\mu(I/I^2) = n$. Let a be a part of a minimal set of generators for $I \bmod I^2$. Let f be a monic polynomial in I. Replace a by $a + f^n$ for suitable n to assume that a is monic. Then $I_1 = I/(a)$ is an ideal in $A_1 = A/(a)$ and $\mu(I_1/I_1^2) = n - 1$. Now A_1 is semilocal as it is integral over R. By Corollary 2.2 $\mu(I_1) = n - 1$. Hence, $\mu(I) = n$. Further, since a appears as a part of a minimal set of generators for I we can add powers of a to the other generators to obtain that each one of them is monic.

COROLLARY 2.4. *If R is a semilocal ring and if I is an ideal containing a monic polynomial in $R[X]$ such that projective dimension of I is finite and I/I^2 is a free $R[X]/I$-module, then I is generated by a regular sequence.*

PROOF. By Ferrand [**1**] or Vasconcelos [**5**] the grade of I equals rank(I/I^2). By Theorem 2.3, $\mu(I) = \mu(I/I^2) =$ grade of I, therefore I is generated by a regular sequence [see **2**, 11.11].

The following example shows that Theorem 2.3 does not extend to ideals that do not contain a monic polynomial.

EXAMPLE 2.5. Let $R = k[[t^2, t^3]]$. Let M be the ideal in $R[X]$ generated by $t^2 - t^3X$ and $1 - t^2X^2$. One easily verifies that M is a maximal ideal of height 1 in $R[X]$. Since $M \cap R = (0)$, $\mu(M/M^2) = 1$. But M cannot be generated by a single element as can be seen without much ado.

The above example involves an element of Pic($R[X]$) which is not extended from R.

EXAMPLE 2.6. Let $D = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1) = \mathbf{R}[x, y]$. Then D is a Dedekind domain. Consider the ideal I generated by $(1 + y)T - x$ and $xT - 1 + y$ in $D[T]$. Then I is in Pic($D[T]$) such that $\mu(I/I^2) = 1$ and $\mu(I) = 2$.

PROOF. Let us first see that I/I^2 is a free $D[T]/I$-module of rank 1. For this, we observe that $(I, 1 + y) = D[T]$. Hence $1 + y$ becomes a unit in $D[T]/I$. Then it is easy to see that $I \bmod I^2$ is generated by the image of $(1 + y)T - x$. Now $\mu(I/I^2) = 1$ implies that I is a projective ideal of rank one and hence an element of

Pic($D[T]$). Suppose that I is principal. Whence it follows that the constant terms of the elements of I generate a principal ideal. But the ideal generated by the constant terms of elements of I is the maximal ideal $(x, y - 1)$ which, being a real point on S^1, requires two generators. Therefore, $\mu(I) = 2$. Furthermore, I as an element of Pic($D[T]$) is extended from R since D is normal.

The following remark shows that neither of the ideals in the examples above contains a monic polynomial.

REMARK 2.7. Let $A = R[T]$ be a polynomial ring. Let I be an ideal containing a monic polynomial in A. If $\mu(I/I^2) = 1$ then $\mu(I) = 1$.

PROOF. The given hypothesis implies that I is a projective ideal of rank 1. Since I contains a monic polynomial, by a theorem of Quillen and Suslin (see [**3**]), I must be principal.

While the presence of a monic polynomial in an ideal I plays such an important role in determining the cardinality of a minimal set of generators for I, one may ask the following: Suppose that an ideal I in a polynomial ring $R[T]$ contains a monic polynomial and $\mu(I) = n$. Is it possible to find a set of n generators for I such that one of them is monic? Curiously enough, the following example illustrates that the answer is no.

EXAMPLE 2.8. Take a Dedekind domain D whose class group has elements of infinite order. To wit, the coordinate ring of the smooth elliptic curve: $Y^2 + Y = X^3 - X$. Choose a prime P in D of infinite order. Then $(P, T) = M$ is a maximal ideal in $D[T]$ such that $\mu(M) = 2$ [**2**, 16.1]. We claim that M cannot be generated by two polynomials such that one of them is monic. Suppose, if possible, that M is generated by f and g in $D[T]$ and that f is monic.

It is a well-known fact that the ideal generated in D by the resultant of f and g is primary to P. Hence P should have finite order-contradiction.

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REFERENCES

1. D. Ferrand, *Suites régulière et intersection complète*, C. R. Acad. Sci. Paris Ser. A **264** (1967), A427–A428.

2. A. V. Geramita and C. Small, *Introduction to homological methods in commutative rings*, Queen's Papers in Pure and Applied Mathematics, Vol. 43 (2nd ed.), Kingston, Ontario, Canada, 1979.

3. T. Y. Lam, *Serre's conjecture*, Lecture Notes in Math., No. 635, Springer-Verlag, Berlin and Heidelberg, 1978.

4. S. Mandal, *On efficient generation of ideals*, Invent. Math. **75** (1984), 59–67.

5. W. Vasconcelos, *Ideals generated by R-sequences*, J. Algebra **6** (1967), 309–316.

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