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STATIONARY POINTS OF PLANE FORMS

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ABSTRACT. We apply formulas for multiple-points of general mappings to enumerate loci of stationary multiple-points of plane forms. This is accomplished by studying the normalization map of the plane form.

1. Introduction. A *plane form* is an l-fold Z in $\mathbf{P}^{l+n}$ ruled by an irreducible one-parameter family of $(l-1)$-planes, such that a general point of the l-fold lies in precisely one plane (or *ruling*). The purpose of this note is to use, in the manner of **[8]**, the stationary multiple-point formulas of **[2]** to give various enumerative formulas for the so-called "stationary points" of Z. Such stationary points are limiting cases of points of Z at which r distinct generating planes meet to those where some of the rulings coincide.

Classical work in this subject was done by James **[5]** and Roth **[14]**. Previous results include the enumerations of the number of pinch-points of a ruled surface in $\mathbf{P}^3$ and of the number of triple-points of a classical 3-fold plane form in $\mathbf{P}^4$ at which two of the generating 2-planes coincide. Both of these results have been derived rigorously using modern techniques (the first in [**6**, §III B, pp. 331–332], for example, and the second in [**13**, 6.5, p. 92]). The benefit to presenting them here is that they are subsumed as special cases of a more general theory and, in this way, we may derive new formulas. In addition, we give some limits on the validity of our results.

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2. Set-up. All schemes and varieties are defined over algebraically closed fields which, for simplicity, we take to be of characteristic zero (the characteristic may in fact remain arbitrary until §4). The basic structure diagram for plane forms Z may be given as follows:

$$\begin{tikzcd} X = \mathbf{P}(E) \arrow[rr, "f"] \arrow[d, "g"'] \arrow[drr, "f'"'] & & Y = \mathbf{P}^{l+n} \\ C & & Z = f(X) \arrow[u, hook] \end{tikzcd}$$

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Here C is a smooth, complete, irreducible curve and E is a locally free sheaf of rank l on C. In order to apply the multiple-point theory of §3, we must also require the following: (1) Assume that the plane form Z spans Y. (2) The total space X of the family, together with the structure maps g and f, corresponds bijectively to a map H: $C \to \mathrm{Gr}_{l-1}(Y)$ of C into the Grassmannian via

$$x \in X \leftrightarrow \big(g(x) \in C \mapsto f\big(g^{-1}g(x)\big) \in \mathrm{Gr}_{l-1}(Y)\big).$$

That is, the fibres of g are mapped as $(l-1)$-planes in Y. We assume that H maps C birationally onto its image in the Grassmannian, so that a general point of Z lies in the image of just one member of X. (3) If Z is l-dimensional, then f': $X \to Z$ is finite. Assume that f' is, in addition, birational.

3. Multiple-point theory (see [**7** and **2**] for details). Let f: $X \to Y$ be a proper map between smooth, irreducible, quasi-projective varieties. Then schemes X_r and maps f_{r-1}: $X_r \to X_{r-1}$ may be constructed inductively according to the following diagram (take f to be separated and $r \geqslant 1$):

$$\begin{array}{ccccc}
 & & X_{r+1} = \mathbf{P}(I) & \hookleftarrow & E_{r+1} = p^{-1}(\Delta) \\
 & & \downarrow p & & \downarrow \\
X_r \overset{p_1}{\leftarrow} & & X_r \times_{X_{r-1}} X_r & \hookleftarrow & \Delta \\
f_{r-1} \downarrow & \square & \downarrow p_2 & & \\
X_{r-1} \underset{f_{r-1}}{\leftarrow} & & X_r & & f_r = (\text{def.})\, p_2 p.
\end{array}$$

Here $X_{r+1} = \mathbf{P}(I)$ is the *residual scheme* of the diagonal Δ in $X_r \times_{X_{r-1}} X_r$, where I is the ideal sheaf of defining the diagonal in the fibered product. (We note that we take $X_0 = Y$, $X_1 = X$, and $f_0 = f$.) The essential feature of X_r is that it parametrizes ordered r-tuples of points of X with the same image under f. The "exceptional locus" $E_r \subset X_r$ parametrizes such r-tuples with the additional requirement that (at least) two of the r points lie "infinitely near" (i.e., determine a tangent direction along the fibre). The exceptional loci need not, in general, be divisors in the derived schemes X_r, although they are schemes of zeros of sections of invertible sheaves.

Given a partition $\mathbf{a} = (a_1, \ldots, a_k)$ of r such that $a_1 \leqslant \cdots \leqslant a_k$ and $\Sigma a_j = r$, we may consider subschemes $T_{\mathbf{a}} \subset X_r$ which parametrize r-tuples "of type $\mathbf{a}$" (see [**2**, 2.2] for the precise definition of $T_{\mathbf{a}}$). That is, a point of $T_{\mathbf{a}}$ is an ordered r-tuple of points of X such that the first a_1 points are infinitely near each other, the next a_2 points are infinitely near, and so on. By a_1 "infinitely near" points, we mean a curvilinear length-a_1 subscheme of the fibre of f having a single point as geometric support. Using the residual intersection theorem (see [**3**, §9.2] or [**7**, §3]) and several other results which hold under operational rational equivalence (e.g., [**2**, 1.10–1.20]), we may define an intersection class $n_{\mathbf{a}}$ and derive a formula for it in $A^{\cdot}(X)$. The class $n_{\mathbf{a}}$ is supported in the locus $N_{\mathbf{a}} = f_1 i_1 \cdots f_{r-1} i_{r-1}(T_{\mathbf{a}})$, where i_s: $X_{s+1} \to X_{s+1}$ is the covering of the map from $X_s \times_{X_{s-1}} X_s$ to itself which switches coordinates (so that, in effect, $f_1 i_1 \cdots f_{r-1} i_{r-1}$ picks off the first coordinates of the r-tuples of $T_{\mathbf{a}}$). Examples of such formulas are (let $n = \operatorname{cod} f = \dim Y - \dim X$ and $c_k = c_k(\nu_f) = [f^*c(Y)/c(X)]_k$ if X and Y are smooth)

$$n_{(2)} = c_{n+1}[X], \tag{3.1}$$

$$n_{(1,2)} = f^*f_*c_{n+1}[X] - 2c_nc_{n+1}[X] - \sum_{j=0}^{n-1} 2^{n-j}c_jc_{2n-j+1}[X], \tag{3.2}$$

$$n_{(3)} = c_{n+1}^2[X] + \sum_{j=0}^{n} 2^{n-j}c_jc_{2n-j+2}[X]. \tag{3.3}$$

$n = 1$ only,

$$\begin{aligned} n_{(1,1,2)} = {} & \left(f^*f_*c_2 - 2c_1c_2 - 2c_3\right)\left(f^*f_*[X] - c_1[X]\right) \\ & - \left(f^*f_*c_1c_2 + c_1f^*f_*c_2 + 2f^*f_*c_3\right)[X] \\ & + \left(4c_1^2c_2 + 4c_2^2 + 12c_1c_3 + 12c_4\right)[X], \end{aligned} \tag{3.4}$$

$$n_{(2,2)} = \left(c_2f^*f_*c_2 - 4c_1c_2^2 - 2c_1^2c_3 - 8c_2c_3 - 10c_1c_4 - 12c_5\right)[X], \tag{3.5}$$

$$\begin{aligned} n_{(1,3)} = {} & \left(f^*f_*c_2^2 + 2f^*f_*c_4 + f^*f_*c_1c_3\right)[X] \\ & - \left(3c_1c_2^2 + c_1^2c_3 + 2c_2c_3 + 4c_1c_4 + 4c_5\right)[X]. \end{aligned} \tag{3.6}$$

For a given partition $\mathbf{a}$ of r, such classes $n_{\mathbf{a}}$ may be defined and such formulas for them are valid when f is $(r; \mathbf{a})$-*generic* (see [**2**, 3.5]). For applications to plane forms, where the map f is between smooth schemes of dimensions l and $l + n$, it suffices to check that the following all have the "correct" (i.e., smallest possible) dimension (or are empty):

$$\begin{aligned} & \dim X_s = l - (s-1)n \quad \text{for } 1 \leqslant s \leqslant r, \\ & \dim E_s = l - (s-1)n - 1 \quad \text{for } 1 \leqslant s \leqslant r, \\ & \dim T_{\mathbf{a}} = l - (r-1)n - (r-k) = l - (r-1)n - \sum_{1}^{k}(a_j - 1). \end{aligned} \tag{3.7}$$

Actually, less restrictive hypotheses suffice, but we will demonstrate that the loci above all have the desired dimensions. We also remark that Ran [**10**], working in a similar, though not identical, context, has in effect reduced the necessary hypotheses to that of requiring only $T_{\mathbf{a}}$ to have correct dimension (or be empty). However, Ran has not made calculations which result in formulas 3.1–3.6 or equivalent versions thereof.

4. Validity. The dimension conditions given in (3.7) needed to apply multiple-point formulas hold for generic projections, as may be seen from Theorem A(i) of [**12**], although Roberts' work does not apply to plane forms. That is, if X is a projective variety, then there is an embedding $X \to \mathbf{P}^N$ for some N such that if $f\colon X \to \mathbf{P}^{l+n}$ $(0 \leqslant n \leqslant l)$ is induced by projection from center L, then, for $1 \leqslant s \leqslant r$, X_s, E_s, and $T_{\mathbf{a}}$ all have the smallest possible dimensions (or are empty) whenever L belongs to a (dense) open subset of the Grassmannian of $(N - l - n - 1)$-planes. The point is that the loci under consideration here are essentially the same as Roberts' $\Sigma_d(\pi;\, q_1,\ldots,q_d)$ loci, whence the result.

Let us consider Kleiman's approach to demonstrating validity of multiple-point formulas [**8**, §3]. His approach leads to an explicitly determined class of plane forms for which (ordinary) multiple-point formulas hold; we extend the results to the stationary case. In particular, these techniques are in keeping with Hilbert's wishes "to establish rigorously and with an exact determination of the limits of their validity those geometrical numbers...by means of the enumerative calculus...," as expressed in his 15th problem (see [**4**]).

LEMMA 4.1. *Let X be a smooth, closed subvariety of dimension l of a projective space $\mathbf{P}^N$ such that X is not contained in any hyperplane. Assume $f\colon X \to \mathbf{P}^{l+n}$ is projection from center L such that* (a) $\bar{S}_2(f) = \varnothing$ *and* (b) *for $s = 2,\ldots,r$ each length-s subscheme of a fibre of f generates an $(s-1)$-plane in $\mathbf{P}^N$. If these conditions are satisfied for L belonging to an open subset of the Grassmannian of $(N-l-n-1)$-planes, then there is a smaller open subset such that if L belongs to it then the conditions in* (3.7) *hold and hence the formula for $n_{\mathbf{a}}$ is valid.* (*Note*, $\bar{S}_2(f) = \{x \in X \mid \dim_x \Omega_f^1 \geqslant 2\}$.)

PROOF. We first note that the structure map $f\colon X \to Y$ for plane forms has no $\bar{S}_2$-singularities because, for $z \in Z$, $f^{-1}(\{z\})$ embeds in $\{z\} \times C$ via (f, g). That the dimensions of X_s and E_s are correct for $1 \leqslant s \leqslant r$ follows immediately from (i) and (ii) of Lemma 3.6 of [**8**]. That $\dim T_{\mathbf{a}} = l - (r-1)n - (r-k)$ (or is empty) follows from the proof of (ii), by noting that the open set $S_{\mathbf{a}} \subset \mathrm{Hilb}_c^r X$ which parametrizes length-r subschemes of X of type $\mathbf{a}$ has codimension $\Sigma(a_j - 1)$ in $\mathrm{Hilb}_c^r X$, the curvilinear Hilbert scheme (see [**9**]). From this it follows (see [**8**] for notation) that the set $d^{-1}(S_{\mathbf{a}} \cap G^{-1}\sigma(L))$ has codimension $r-k$ in X_r and is dense in $T_{\mathbf{a}}$.

PROPOSITION 4.2 (SEE [**8**, 3.7]). *Consider the set-up for plane forms. Suppose f: $X \to \mathbf{P}^{l+n}$ factors as an embedding into some $\mathbf{P}^N$ followed by a central projection and, in addition, for $s = 2,\ldots,r$, each length-s subscheme of X generates an $(s-1)$-plane of $\mathbf{P}^N$ whenever g restricted to the subscheme is an embedding. Then the formula for $n_{\mathbf{a}}$ is valid and* $\deg n_{\mathbf{a}}$ *is $p(\mathbf{a})$ times the degree of the locus $N_{\mathbf{a}}$, which is $q_1 \cdot p(\mathbf{a})$ times the degree of the closure of the set of points of Z at which r branches meet, but the rulings coincide according to the partition* $\mathbf{a}$ *of r.* (*See* [**2**, §3] *for definitions of q_1 and $p(\mathbf{a})$.*)

Characteristic zero is needed in Proposition 4.2 for the simple description given for $\deg n_{\mathbf{a}}$, but it is not needed for the validity of the formula for the class $n_{\mathbf{a}}$. Kleiman specifies a class of plane forms [**8**, 3.8] for which, as will be evident in the sequel, the formula for $n_{\mathbf{a}}$ is valid and has the desired meaning.

KLEIMAN'S CLASS. Assume that C has a fixed embedding in some projective space $\mathbf{P}(V_1)$. Assume E is such that $E(-1)$ is generated by global sections and fix V_2, a space of generating sections. Let V be a general $(l+n+1)$-dimensional subspace of $V_1 \otimes V_2$. We have then that f factors as

$$X \hookrightarrow \mathbf{P}(V_1) \times \mathbf{P}(V_2) \hookrightarrow \mathbf{P}(V_1 \otimes V_2) = \mathbf{P}^N \to Y = \mathbf{P}(V)$$

where the last map is central projection. It follows that the map $H\colon C \to \mathrm{Gr}_{l-1}(Y)$ defined in §2 is an embedding.

As an easy consequence of Proposition 4.2, we have

THEOREM 4.3. *Assume that the plane form set-up is as given by Kleiman's class. If X is r-twisted in $\mathbf{P}^N$ (meaning any length-r subscheme of C spans an $(r-1)$-plane in $\mathbf{P}(V_1)$), then the conclusions of Proposition* 4.2 *hold.*

5. Computations. Having established the validity and meaningfulness of our multiple-point formulas, we may apply the results to plane forms in the class specified in §4. First we state some results for the intersection theory of the set-up of §2 and then apply these to several examples.

Define the following classes on $A(X)$:

$$h = f^*c_1(\mathcal{O}_Y(1)) = c_1(\mathcal{O}_X(1)), \qquad e = g^*c_1(E), \qquad t = g^*c_1(C).$$

These classes satisfy the following relations:

$$e^2 = t^2 = et = 0, \qquad eh^{l-1} = h^l. \tag{5.1}$$

We also have the following results:

LEMMA 5.2 [**8**, 2.4(ii)]. *Let α be a q-cycle on X. Then*

$$f^*f_*\alpha = \left(\int_X \alpha \cdot h^q\right) h^{l+n-q}.$$

Alternatively, if β is a codimension-k cycle, we have

$$f^*f_*\beta = \left(\int_X \beta \cdot h^{l-k}\right) h^{n+k}.$$

LEMMA 5.3 [**8**, 2.4(i)]. (a) $\int_X h^l = \int_X eh^{l-1} = d =$ *degree of Z in $Y = \mathbf{P}^{l+n}$.*
(b) $\int_X th^{l-1} = 2 - 2p$, *where p = genus of C.*

PROPOSITION 5.4 [**8**, 2.3.2]. *The total Chern class of the virtual normal bundle ν_f of f is given by*

$$c(\nu_f) = \frac{f^*c(Y)}{c(X)} = 1 + \sum_{i\geq 1}\left[\binom{n+1}{i}h - \binom{n+1}{i-1}t + \binom{n}{i-1}e\right]h^{i-1}.$$

For ease of reference, the formula for $c(\nu_f)$ when $n = 1$ (the classical case of a primal) is

$$c_1(\nu_f) = 2h - t + e, \qquad c_2(\nu_f) = h^2 - 2th + eh,$$
$$c_3(\nu_f) = -th^2, \qquad c_j(\nu_f) = 0, \quad j \geq 4.$$

EXAMPLE 5.5. Let Z be a ruled surface in $\mathbf{P}^3$, so $l = 2$, $n = 1$. Then

$$n_{(2)} = c_2[X] = h^2 - 2th + eh.$$

Thus, by (5.1) and Lemma 5.3, $\int_X n_{(2)} = 2(d + 2p - 2) = \#$pinch-points of $Z = 2\#$pinch-points if char $k = 2$ (see [**11**, §5, pp. 161–163]). Note that this result is also derived in [**6**, III, 23, p. 332] and [**1**, p. 166].

EXAMPLE 5.6. Let Z be a "classical" plane form in $\mathbf{P}^4$, so that $l = 3$, $n = 1$. Then the degree of $n_{(2)}$ is the degree of the curve of pinch-points which is given by the formula in Example 5.5. We also have

$$n_{(1,2)} = \left(f^*f_*c_2 - 2c_1c_2 - 2c_3\right)[X] = (2d + 4p - 8)h^3 + 12th^2 - 6eh^2.$$

Then, by Lemma 5.3 and Theorem 4.3 we conclude that $\int_X n_{(1,2)} = 2d^2 - 14d + 4pd - 24p + 24 =$ #stationary triple-points of Z at which three branches meet, but two of the generating planes coincide. This is Roth's result [**14**, formula 12, p. 122] and was rederived in [**13**, p. 93] using essentially the same formula as in (3.2), though this general result was calculated using a different point of view.

EXAMPLE 5.7. Let Z be a primal contained in $\mathbf{P}^5$, so that $l = 4$, $n = 1$. From (3.3) with $n = 1$, we have

$$n_{(3)} = \left(c_2^2 + 2c_4 + c_1c_2\right)[X].$$

Thus $n_{(3)} = h^4 - 6th^3 + 2eh^3$, so that $\int_X n_{(3)} = 3d + 2p - 12 =$ #"super pinch-points" (i.e., triple points at which all three of the rulings coincide). Similarly, using (3.5), we find

$$\begin{aligned} n_{(1,1,2)} = {} & (2d^2 - 22d + 4pd - 32p - 60)h^4 \\ & + (14d + 4p - 120)th^3 + (-8d - 4p + 60)eh^3. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2}\int_X n_{(1,1,2)} &= d(d^2 - 15d + 2pd - 18p - 60) + (2 - 2p)(7d + 2p - 60) \\ &= \#\text{stationary quadruple-points.} \end{aligned}$$

EXAMPLE 5.8. Let $Z^5 \subset \mathbf{P}^6$. Then

$$n_{(2,2)} = (2d + 4p - 12)h^5 + (-4d - 8p + 60)th^4 + (2d + 4p - 24)eh^4,$$

so

$$\begin{aligned} \frac{1}{2}\int_X n_{(2,2)} &= d(2d + 4p - 18) + (2 - 2p)(-2d - 4p + 30) \\ &= \#\text{quadruple-points of type } (2,2). \end{aligned}$$

Similarly,

$$n_{(1,3)} = (3d + 2p - 18)h^5 + 45th^4 - 15eh^4,$$

so

$$\begin{aligned} \int_X n_{(1,3)} &= 3((d - 5)(d - 6) + 2p(2d - 15)) \\ &= \#\text{quadruple-points of type } (1,3). \end{aligned}$$

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