PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 3, November 1985

ON G-SYSTEMS AND G-GRADED RINGS

P. GRZESZCZUK

ABSTRACT. Rings graded by finite groups and homomorphic images of such rings are studied. Obtained results concern finiteness conditions and radicals.

Introduction. Our aim in this paper is the study of rings graded by finite groups. To obtain some results we need information on homomorphic image of a graded ring (cf. Theorem 5). The proofs of other results work in this more general situation as well. For these reasons the paper concerns G-systems, defined as follows.

Let G be a finite group with identity e. A ring R is said to be G-system if $R = \sum_{g \in G} R_g$, where R_g are such additive subgroups of R that $R_g R_h \subseteq R_{gh}$ for all g, $h \in G$. If for all $g, h \in G$, $R_g R_h = R_{gh}$, R is called [**3, 4, 7**] the Clifford system.

Certainly any G-graded ring is a G-system and the class of G-systems is homomorphically closed, while G-graded rings do not necessarily have this property. It is easy to check that every G-system is a homomorphic image of a G-graded ring.

In this paper we prove that a "Clifford type" theorem holds for every G-system R. Namely, we show that simple R-modules (R-bimodules) are completely reducible R_e-modules (R_e-bimodules) and that $J(R_e) = J(R) \cap R_e$, $U(R_e) \subseteq U(R)$, where $J(\text{-})$, $U(\text{-})$ denote the Jacobson and the Brown-McCoy radical, respectively. Using this method we obtain, in particular, a quite different proof from those given by M. Cohen and S. Montgomery [**2**] and M. Van den Bergh [**6**] of Bergman's conjecture. We also prove that the R-module M is Noetherian if and only if M is Noetherian as an R_e-module.

1. We start with

THEOREM 1. *If the G-system $R = \sum_{g \in G} R_g$ has unity* 1, *then* $1 \in R_e$.

PROOF. Define for any nonempty subset S of G, $R_S = \sum_{s \in S} R_s$. It is clear that for all $\varnothing \neq S$, $T \subseteq G$, $R_S R_T \subseteq R_{ST}$. We prove by induction on $|G \setminus S|$ that if $e \in S$ then $1 \in R_S$. If $S = G$ then $1 \in R_G = R_S$. Assume the result is true for subsets of cardinality $> |S|$. Let $x \in G \setminus S$; then $|S \cup \{x\}| > |S|$ and $|x^{-1}S \cup \{e\}| > |S|$. Hence by induction assumption $1 \in R_S + R_x$ and $1 \in R_{x^{-1}S} + R_e$. That is, there exist $\alpha(S) \in R_S$ and $\alpha(x) \in R_x$ so that $1 + \alpha(S) = \alpha(x)$ and $\beta(x^{-1}S) \in R_{x^{-1}S}$, $\beta(e) \in R_e$ so that $1 + \beta(e) = \beta(x^{-1}S)$. However, $(1 + \alpha(S))(1 + \beta(e)) = \alpha(x)\beta(x^{-1}S) \in R_x R_{x^{-1}S} \subseteq R_S$. Thus, since $e \in S$, $(1 + \alpha(S))(1 + \beta(e)) = 1 + \gamma(S)$, where $\gamma(S) \in R_S$. Hence, for some $\delta(S) \in R_S$, $1 + \gamma(S) = \delta(S)$. In particular, $1 \in R_S$. Thus, since $S = \{e\}$ satisfies the hypothesis, $1 \in R_e$.

Received by the editors October 26, 1984.
1980 *Mathematics Subject Classification*. Primary 16A03, 16A21.

REMARK 1. Obviously the notion of G-system can be extended to infinite G. However, Theorem 1 does not hold in this case. Namely, let A be the ring of all rational numbers of the form $2n/(2m+1)$, where n, m are integers and let $R = A[x]$ be the polynomial ring. There is a Z-gradation on R such that $R_0 = A$. The homomorphism $f\colon R \to Q$ given by $f(w(x)) = w(\frac{1}{2})$ maps R onto the field Q of rational numbers. Thus Q is a Z-system with unity 1 and $1 \notin A = Q_0$.

As a consequence of Theorem 1 we obtain

COROLLARY 1. *Let $R = \sum_{g\in G} R_g$ be a G-system with unity* 1. *Then*

(a) *if I is a right (left) ideal of R_e, then $IR = R$ ($RI = R$) if and only if $I = R_e$,*

(b) *an element $x \in R_e$ is right (left) invertible in R if and only if x is right (left) invertible in R_e,*

(c) $J(R) \cap R_e \subseteq J(R_e)$.

PROOF. (a) If $I = R_e$, then by Theorem 1, $1 \in I$ and $IR = R$. Conversely, let $R = IR = \sum_{g\in G} IR_g$. Since, for all $g, h \in G$, $(IR_g)(IR_h) = I(R_g IR_h) \subseteq IR_{gh}$, $IR = \sum_{g\in G} IR_g$ is a G-system with unity. By Theorem 1, $1 \in IR_e \subseteq I$, so $I = R_e$.

(b) Let $x \in R_e$ be right invertible in R. Consider a right ideal $I = xR_e$ of R_e. Since $IR = R$, by (a) we have $I = R_e$. Therefore, $xx' = 1$ for some $x' \in R_e$.

(c) By (b), $J(R) \cap R_e$ is a quasi-regular ideal of R, so $J(R) \cap R_e \subseteq J(R_e)$.

REMARK 2. Corollary 1(c) holds also for G-systems without unity. Namely, let $R = \sum_{g\in G} R_g$ be any G-systems. Defining on the additive group $\overline{R} = R \times Z$ multiplication by $(x, m)(y, n) = (xy + nx + my, mn)$, we obtain a natural extension of R to a ring with unity. Moreover, the ring $\overline{R}$ has a structure of G-system $\overline{R} = \sum_{g\in G}\overline{R}_g$, where $\overline{R}_e = R_e \times Z$ and, for $g \neq e$, $\overline{R}_g = R_g \times \{0\}$. It is clear that $J(R) = J(\overline{R})$ and $J(R_e) = J(\overline{R}_e)$. Thus, $J(R) \cap R_e = J(\overline{R}) \cap \overline{R}_e \subseteq J(\overline{R}_e) = J(R_e)$.

The following lemma is in fact the crucial step in the proof of a "Clifford type" theorem for G-systems.

LEMMA 1. *Let M be a right module over G-system $R = \sum_{g\in G} R_g$ and let $0 \neq M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ be a chain of R_e-submodules of M_{R_e} such that $\bigcup_{n=1}^{\infty} M_n$ is essential in M_{R_e}. Then for some $m \geqslant 1$, M_m contains a nonzero R-submodule of M.*

PROOF. Let us observe that using the procedure of Remark 2, we can reduce the proof to G-systems with unity. Now we shall prove by induction on $k = 1, 2, \ldots, |G|$ that there exist a subset $H_k \subseteq G$ and a nonzero element $m_k \in M$ such that

1°. $e \in H_k$,

2°. $|H_k| = k$,

3°. $0 \neq \sum_{h\in H_k} m_k R_h \subseteq M_{s(k)}$ for some $s(k) \geqslant 1$.

For $k = 1$ we put $H_1 = \{e\}$, m_1 any nonzero element of M_1 and $s(1) = 1$. Let $|G| > k \geqslant 1$ and m_k, H_k satisfy 1°–3°. Consider an element $g \notin H_k$. If $m_k R_g = 0$, then $m_{k+1} = m_k$, $H_{k+1} = H_k \cup \{g\}$ satisfy 1°–3°. If $m_k R_g \neq 0$, then by essentiality of $\bigcup_{n=1}^{\infty} M_n$, there exists $r_g \in R_g$ such that $0 \neq m_k r_g \in M_t$ for some $t \geqslant 1$. Let

$m_{k+1} = m_k r_g$, $H_{k+1} = g^{-1}H_k \cup \{e\}$ and $s(k+1) = \max\{s(k), t\}$. Clearly, $|H_{k+1}| = k+1$ and, since $r_g R_{g^{-1}h} \subseteq R_g R_{g^{-1}h} \subseteq R_h$,

$$0 \neq \sum_{h \in H_{k+1}} m_{k+1} R_h = m_{k+1} R_e + \sum_{h \in H_k} m_k r_g R_{g^{-1}h}$$

$$\subseteq M_t + \sum_{h \in H_k} m_k R_h \subseteq M_t + M_{s(k)} = M_{s(k+1)}.$$

Applying 3° to $k = |G|$ we obtain that, for some $m \geqslant 1$, M_m contains a nonzero R-submodule of M_R.

Now we can prove a "Clifford type" theorem for G-systems.

THEOREM 2. *If M is a simple right R-module, then $M_R = M_1 \oplus \cdots \oplus M_k$ is a direct sum of $k \leqslant |G|$ simple R_e-modules.*

PROOF. Let us observe that any nonzero R_e-submodule N of M is its direct summand. Indeed, let K be a maximal with respect to $N \cap K = 0$ submodule of M_{R_e}. Then $N \oplus K$ is an essential submodule of M_{R_e}. Since M is a simple R-module, by Lemma 1 we obtain that $N \oplus K = M$.

Thus every nonzero R_e-submodule of M contains a simple R_e-module. Let H be a subset of G containing e, of maximal cardinality, such that, for some $m \in M$, $\sum_{h \in H} mR_h \neq 0$ and, for all $h \in H$, $mR_h = 0$ or mR_h is a simple R_e-module. We claim that $H = G$. Indeed, if $g \in G \setminus H$, then $mR_g \neq 0$. Let $\overline{m}R_e$ be a simple R_e-submodule of mR_g. Then for $h \in H$, $\overline{m}R_{g^{-1}h} \subseteq mR_g R_{g^{-1}h} \subseteq mR_h$. Thus if $\bar{h} \in g^{-1}H \cup \{e\}$, then $\overline{m}R_{\bar{h}} = 0$ or $\overline{m}R_{\bar{h}}$ is a simple R_e-module. This contradicts maximality of H and proves the claim.

Therefore, M is a sum of $k \leqslant |G|$ simple R_e-modules.

Theorem 2 and Remark 2 imply immediately

COROLLARY 2. *If $R = \sum_{g \in G} R_g$ is a G-system, then*

(a) *for any right R-module M, $J(M_{R_e}) \subseteq J(M_R)$,*

(b) *for any right R-module M, $\operatorname{Soc}(M_R) \subseteq \operatorname{Soc}(M_{R_e})$.where $\operatorname{Soc}(\text{-})$ denotes the socle,*

(c) (*cf.* [**2**]) $J(R_e) = J(R) \cap R_e$.

The graded Jacobson radical $J_G(R)$ of a G-graded ring $R = \bigoplus_{g \in G} R_g$ is defined in [**1**] as the ideal of R satisfying the following equivalent conditions:

1. $J_G(R)$ is the intersection of all maximal graded right ideals of R.
2. $J_G(R)$ is the largest graded ideal I of R such that $I \cap R_e$ is a quasi-regular ideal of R_e.

THEOREM 3. (*cf.* [**2, 6**]). *For every ring R graded by finite group G, $J_G(R) \subseteq J(R)$.*

PROOF. Obviously, $I = RJ(R_e)R$ is a graded ideal of R. By Corollary 2(c), $I \subseteq J(R)$ and $J(R_e) \subseteq I \cap R_e \subseteq J(R) \cap R_e = J(R_e)$. Thus, $I \cap R_e = J(R_e)$, so $I \subseteq J_G(R)$. Consider the ring $S = J_G(R)/I$. Clearly, S is a G-graded ring and its identity component $S_e = 0$. Thus by [**1**], $S^{|G|} = 0$. Therefore, $J_G(R)$ is a J-radical ideal of R and $J_G(R) \subseteq J(R)$.

We close this section by

THEOREM 4. *Let M be a right module over G-system $R = \sum_{g \in G} R_g$. Then M_R is Noetherian if and only if M_{R_e} is Noetherian.*

PROOF. Suppose that there exists Noetherian R-module M which is not Noetherian as R_e-module. Using Noetherian induction we may assume that, for each nonzero R-submodule N, the module $(M/N)_{R_e}$ is Noetherian.

Let $X_1 \subsetneq X_2 \subsetneq \cdots \subseteq M_{R_e}$ be a strictly ascending chain of R_e-submodules and let Y be an R_e-submodule of M maximal with respect to $(\bigcup_{n=1} X_n) \cap Y = 0$. Now we have a strictly ascending chain $X_1 \oplus Y \subsetneq X_2 \oplus Y \subsetneq \cdots$ of R_e-submodules of M such that $\bigcup_{n=1}^{\infty} X_n \oplus Y = (\bigcup_{n=1}^{\infty} X_n) \oplus Y$ is essential in M_{R_e}. By Lemma 1, for some $m \geqslant 1$, the module $X_m \oplus Y$ contains a nonzero, say N, R-submodule. Therefore, in $(M/N)_{R_e}$, we have a strictly ascending chain

$$X_{m+1} \oplus Y/N \subsetneq X_{m+2} \oplus Y/N \subsetneq \cdots \subseteq M/N,$$

a contradiction.

The converse is clear.

REMARK 3. Theorem 4 implies that if the ring R is right Noetherian then the ring R_e is right Noetherian. The converse is not true in general. Consider the matrix ring

$$R = \begin{pmatrix} k & A \\ 0 & k \end{pmatrix},$$

where k is a field and A an arbitrary infinite-dimensional k-algebra. Putting

$$R_0 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

we obtain a C_2-gradation on R. Clearly, R_0 is right Noetherian but R contains a strictly ascending chain of right ideals of R, e.g.

$$\begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix} \subsetneq \begin{pmatrix} 0 & V_2 \\ 0 & 0 \end{pmatrix} \subsetneq \cdots,$$

where $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq A$ is a chain of k-subspaces of A.

2. In this section we show that some of the previous results can be applied to bimodules. Moreover, we obtain a link between both Brown-McCoy radicals and prime ideals of R_e and R. Now, G-systems do not necessarily have a unity element.

Let us observe that the ring $S = R^0 \otimes_Z R$ (R^0 is the opposite ring of R) has a structure of the $G \times G$-system. For $(g, h) \in G \times G$ we make $S_{(g,h)}$ the additive subgroup of S generated by $r_g^0 \otimes r_h$, where $r_g^0 \in R_g^0$, $r_h \in R_h$. Clearly, an R-sub-bimodule, or an R_e-subbimodule of an R-bimodule V, is simply a right submodule over S or $S_{(e,e)}$, respectively. Thus, by Theorem 2, we have

COROLLARY 3. *If V is a simple R-bimodule, then ${}_{R_e}V_{R_e} = V_1 \oplus \cdots \oplus V_k$ is a direct sum of $k \leqslant |G|^2$ simple R_e-bimodules.*

COROLLARY 4. *If J is a maximal ideal of the G-system $R = \sum_{g \in G} R_g$, then $J \cap R_e$ is a finite intersection of $k \leqslant |G|^2$ maximal ideals of R_e.*

PROOF. Consider the G-system $\overline{R} = R/J$. Obviously, $\overline{R}$ is a simple R-bimodule. By Corollary 3, there exist simple R_e-subbimodules $\overline{V}_1, \ldots, \overline{V}_k$ ($k \leqslant |G|^2$) such that ${}_{R_e}\overline{R}_{R_e} = \overline{V}_1 \oplus \cdots \oplus \overline{V}_k$. Thus, for $\overline{W}_i = \overline{V}_1 \oplus \cdots \oplus \overline{V}_{i-1} \oplus \overline{V}_{i+1} \oplus \cdots \oplus \overline{V}_k$, $\overline{O} = \bigcap_{i=1}^{k} \overline{W}_i$ and $\overline{W}_i$ are maximal R_e-subbimodules of ${}_{R_e}\overline{R}_{R_e}$. Therefore, the ideal J is an intersection of maximal R_e-subbimodules $W_1, \ldots, W_k$ of R and $W_i \cap R_e$ ($i = 1, 2, \ldots, k$) are maximal ideals of R_e.

Let, for a ring R, $U(R)$ denote the Brown-McCoy radical of R, i.e. the intersection of all ideals I of R such that R/I is a simple ring with unity.

THEOREM 5. *For every G-system* $R = \sum_{g\in G} R_g$, $U(R_e) \subseteq U(R)$.

PROOF. Let I be a maximal ideal of R such that $\bar{R} = R/I$ is a simple ring with unity. Since $\bar{R}$ is a G-system, by Theorem 1 the ring $\bar{R} = R_e/I \cap R_e$ has unity. By Corollary 4, there exist maximal ideals $I_1, \ldots, I_k$ of R_e such that $I \cap R_e = I_1 \cap \cdots \cap I_k$. Obviously, for $i = 1, \ldots, k$, R_e/I_i are simple rings with unity, so $U(R_e) \subseteq U(R)$.

We now give an example of a C_2-graded ring $R = R_0 \oplus R_1$ with $U(R_e) \neq U(R) \cap R_e$.

EXAMPLE. Let V be an infinite-dimensional vector space over a field k and let V_0, V_1 be subspaces of k such that $\dim_k V_0 = 1$ and $V_0 \oplus V_1 = V$. Consider the ring R of all linear transformations of finite rank of V. R is a simple ring without unity, so $U(R) = R$. On R we define a C_2-gradation putting

$$R_0 = \{ f \in R \mid f(V_0) \subseteq V_0, f(V_1) \subseteq V_1 \},$$
$$R_1 = \{ f \in R \mid f(V_0) \subseteq V_1, f(V_1) \subseteq V_0 \}.$$

Let us observe that R_0 is isomorphic to the ring $k \oplus R$. Hence R_0 is not a U-radical ring.

In [5] Passman proved that if R is a prime ring then there exist sets Y, Z such that the power series-polynomial ring $R^* = (R\langle\langle Y \rangle\rangle)\langle Z \rangle$ is primitive. Clearly, if $R = \sum_{g\in G} R_g$ is a G-system, then $R^* = \sum_{g\in G} R_g^*$ is a G-system, where R_g^* denotes the set of all power series-polynomials with coefficients in R_g. Using Passman's method (see [**5**, Theorem 3.1]) and Theorem 2 we obtain

THEOREM 6 (CF. [**2**]). *If P is a prime ideal of R, then $P \cap R_e$ is a finite intersection of $k \leqslant |G|$ prime ideals of R_e.*

As an immediate consequence of Theorem 6 we have

COROLLARY 5 (CF. [**2**]). $B(R_e) = B(R) \cap R_e$, *where $B(\text{-})$ denotes prime radical.*

ACKNOWLEDGMENT. I am deeply grateful to Dr. E. R. Puczyłowski for his valuable suggestions and for stimulating discussions. I would also like to thank the editor for suggestions making the proof of Theorem 1 clearer.

REFERENCES

1. G. Bergman, *On Jacobson radical of graded rings*, Preprint.
2. M. Cohen and S. Montgomery, *Group-graded rings, smash products and group actions*, Trans. Amer. Math. Soc. **282** (1985), 237–258.
3. E. C. Dade, *Compounding Clifford's theory*, Ann. of Math. (2) **91** (1970), 236–290.
4. C. Năstăsescu, *Strongly graded rings of finite groups*, Comm. Algebra **11** (1983), 1033–1071.
5. D. Passman, *Prime ideals in normalizing extensions*, J. Algebra (2) **73** (1981), 556–572.
6. M. Van den Bergh, *On a Theorem of S. Montgomery and M. Cohen*, Preprint.
7. F. Van Oystaeyen, *On Clifford systems and generalized crossed products*, Preprint.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, BIAŁYSTOK DIVISION, AKADEMICKA 2, 15–267 BIAŁYSTOK, POLAND