PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95. Number 3. November 1985

A REMARK ON THE STRUCTURE OF ABSOLUTE GALOIS GROUPS

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ABSTRACT. Let the field F contain all p-power roots of unity for some prime p and suppose that the absolute Galois group G of F is a one-relator pro-p group. We use Merkurjev-Suslin's theorem on the power norm residue symbol to show that G is an extension of a Demushkin group by a free pro-p group.

Let F be a field, F_s its separable closure, and $G = \operatorname{Gal}(F_s/F)$ its absolute Galois group. Let n be an integer not divisible by $\operatorname{char}(F)$ and denote by μ_n the G-module of nth roots of unity. Merkurjev and Suslin [1] have shown that the power norm residue symbol $K_2(F)/nK_2(F) \to H^2(G, \mu_n^{\otimes 2})$ is an isomorphism. It is a natural question to ask what bearing this theorem has on the structure of G. The present note is based on the fact that the symbol above factors through the homomorphism $H^1(G, \mu_n)^{\otimes 2} \to H^2(G, \mu_n^{\otimes 2})$, induced by the cup product, which hence is surjective. To be modest, we specialize to Sylow subgroups of G, i.e., we let G itself be a pro-p group. Using well-known properties of such groups we then derive some information in the case where G has exactly one relation and F contains all p-power roots of unity.

The notation will be standard: subgroups of a pro-p group G are understood to be closed; the Frattini subgroup is $G^* = G^pG_2$, G_2 denoting the commutator subgroup; subgroups generated by commutators are written as $[X, Y]$; cohomology groups with coefficients being $\mathbf{Z}/(p)$ are denoted by $H^i(G)$.

THEOREM. *Let F be a field with separable closure F_s and absolute Galois group $G = \operatorname{Gal}(F_s/F)$. Suppose that G is a finitely generated one-relator pro-p group where the prime p is unequal to* $\operatorname{char}(F)$ *and F contains all p-power roots of unity.*

Then there is a normal subgroup N of G which is pro-p free such that G/N is a Demushkin group and the inflation map $H^2(S/N, \mathbf{Z}/(p^n)) \to H^2(S, \mathbf{Z}/(p^n))$ is an isomorphism for every subgroup S of G containing N, and for all integers n.

PROOF. (1) We begin by constructing a normal subgroup N of G such that, setting $\overline{G} = G/N$, the inflation map $H^2(\overline{G}) \to H^2(G)$ is an isomorphism and the cup product

$$H^1(\overline{G}) \times H^1(\overline{G}) \to H^2(\overline{G})$$

is nondegenerate.

Received by the editors December 6, 1984.
1980 *Mathematics Subject Classification*. Primary 12G05.

Let X be the subgroup of G containing G^* such that $H^1(G/X)$ is the radical of the cup product map $H^1(G) \times H^1(G) \overset{\cup}{\to} H^2(G)$. Choose a second subgroup Y of G containing G^* so that $G/G^* = X/G^* \times Y/G^*$. Compactness of G then yields a subgroup Z of Y which is minimal with respect to $Y = ZG^*$. This implies that $Z \cap G^* = Z^*$. Set $N = \langle\langle Z \rangle\rangle$, the normal subgroup of G generated by Z. We then have $N = Z[N, G]$, hence $N \cap G^* = Z^*[N, G] \leqslant N^p[N, G]$, and so $N \cap G^* = N^p[N, G]$. Therefore, the restriction map $H^1(G) \to H^1(N)^G$ is onto and so the five term cohomology sequence makes $H^2(\overline{G}) \to H^2(G)$ one-to-one. On the other hand, $H^1(G) = H^1(G/X) \oplus H^1(\overline{G})$. So, by Merkurjev-Suslin's result, $H^2(G)$ is generated by cup products from $H^1(\overline{G})$ and the cup product restricted to this latter space is nondegenerate. The claim now follows from the following commutative diagram.

$$\begin{tikzcd}
H^1(\overline{G}) \times H^1(\overline{G}) \arrow[rr, "\cup"] \arrow[dr, "\cup"'] & & H^2(G) \\
& H^2(\overline{G}) \arrow[ur] &
\end{tikzcd}$$

(2) We have not yet made use of the hypothesis dim $H^2(G) = 1$. This, together with the result of (1) makes $\overline{G}$ a Demushkin group (cf. [**2**, no. 3]). In particular, $\overline{G}$ has cohomological dimension 2 [**2**, no. 9]; the exceptional case $\overline{G} = \mathbf{Z}/(2)$ being ruled out by what follows. In fact, the abelianized group $\overline{G}_{\text{ab.}}$ is torsion free. This can be seen as follows: $G_{\text{ab.}}$ is torsion free because F contains all p-power roots of unity. Applying cohomology to $0 \to \mathbf{Z}/(p) \to P \overset{p}{\to} P \to 0$ where $P = \mathbf{Q}_p/\mathbf{Z}_p$, and the isomorphism of (1) yield the following commutative diagram with exact rows.

$$\begin{tikzcd}
H^1(\overline{G}, P) \arrow[r, "p"] \arrow[d] & H^1(\overline{G}, P) \arrow[r] \arrow[d] & H^2(\overline{G}) \arrow[d, "\wr"] \\
H^1(G, P) \arrow[r, "p"] & H^1(G, P) \arrow[r, "0"] & H^2(G)
\end{tikzcd}$$

Thus, $H^1(\overline{G}, P)$ is divisible as well and $\overline{G}_{\text{ab.}}$ is torsion free.

We shall also need the fact that the inflation maps $H^2(\overline{G}, R_n) \to H^2(G, R_n)$ are isomorphisms for all $R_n = \mathbf{Z}/(p^n)$. This follows by induction from the cohomology sequences for $0 \to R_1 \to R_{n+1} \to R_n \to 0$, using $\mathrm{cd}(\overline{G}) = 2$ and torsion freeness of G_{ab}.

(3) The rest of the theorem is proved by considering the Brauer groups involved. We want to show that N is pro-p free and that the $\overline{G}$-module $A = \overline{F}^\times/\mu$, where $\overline{F} = F_s^N$ and μ is the group of all p-power roots of unity, satisfies $H^k(\overline{G}, A) = 0$ for $k = 1, 2$. We have the following commutative diagram whose upper row is exact as well as its left column which derives from $1 \to \mu \overset{j}{\to} \overline{F}^\times \to A \to 1$, taking into

account Hilbert's Theorem 90 and $\mathrm{cd}(\overline{G}) = 2$:

$$\begin{tikzcd}
 & 0 & & & \\
 & H^2(\overline{G}, A) \arrow[u] & & & \\
0 \arrow[r] & H^2(\overline{G}, \overline{F}^{\times}) \arrow[u] \arrow[r, "\mathrm{inf}"] & H^2(G, F_s^{\times}) \arrow[r] & H^2(N, F_s^{\times})^G \arrow[r, "tg"] & H^3(\overline{G}, \overline{F}^{\times}) \\
 & H^2(\overline{G}, \mu) \arrow[u, "j_*"] \arrow[r, "i"] & H^2(G, \mu) \arrow[u, "\wr"] & H^2(N, \mu)^G \arrow[u, "\wr"] \arrow[r] & H^3(\overline{G}, \mu) = 0 \arrow[u] \\
 & H^1(\overline{G}, A) \arrow[u] & & & \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

The two vertical isomorphisms come from the usual description of Brauer groups using roots of unity. Thus the transgression map tg is zero. By (3), the inflation map i is an isomorphism. Hence so are inf and j_*. It follows that $H^k(\overline{G}, A) = 0$ for $k = 1, 2$ and that $H^2(N, \mu)^G = 0$. Since G is pro-p and $N_{\mathrm{ab.}}$ is torsion free, $H^2(N) = 0$ and N is pro-p free.

(4) The last statement of the theorem needs only to be shown for open subgroups S of G. Put $\overline{S} = S/N$. Since $S_{\mathrm{ab.}}$ and $\overline{S}_{\mathrm{ab.}}$ are torsion free, it suffices to show that $H^2(\overline{S}, \mu) \simeq H^2(S, \mu)$. Using $H^k(\overline{G}, A) = 0$ for $k = 1, 2$ and p-torsion freeness of A, the method of the criterion for cohomological triviality [**3**, p.150, Théorème 6] yields that $H^2(\overline{S}, A) = 0$. We have a diagram analogous to that in (3)

$$\begin{tikzcd}
H^2(\overline{S}, \overline{F}) \arrow[r] & H^2(S, F_s) \\
H^2(\overline{S}, \mu) \arrow[u, "j_*"] \arrow[r, "i"] & H^2(S, \mu) \arrow[u]
\end{tikzcd}$$

where j_* is onto. So it remains to verify that i is injective which in turn needs only to be shown for $H^2(\overline{S}) \xrightarrow{i'} H^2(S)$ because $\overline{S}$ and S abelianized are torsion free. We use corestriction to do so. The diagram

$$\begin{tikzcd}
H^2(\overline{S}) \arrow[r, "i'"] \arrow[d, "\overline{\mathrm{cor}}"'] & H^2(S) \arrow[d, "\mathrm{cor}"] \\
H^2(\overline{G}) \arrow[r, "\sim"] & H^2(G)
\end{tikzcd}$$

commutes because $N \leqslant S \leqslant G$. The corestriction map $\overline{\mathrm{cor}}$ is onto because $\mathrm{cd}(\overline{G}) = 2$ and hence is an isomorphism because both $\overline{G}$ and $\overline{S}$ are one-relator groups [**2**, no. 9]. □

Let us close with a question. The cohomological condition given in the theorem is somewhat obscure. Does it imply that G is of the form $G_0 \amalg \bar{G}$, the free pro-p product of a free pro-p group G_0 and a Demushkin group $\bar{G}$?

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