PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 3, November 1985

POSITIVE DEFINITE BOUNDED MATRICES AND A CHARACTERIZATION OF AMENABLE GROUPS[1]

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ABSTRACT. We show that a discrete group G is amenable iff the Herz-Schur multiplier algebra $B_2(G)$ coincides with the Fourier-Stieltjes algebra $B(G)$.

1. Introduction and notation. Let X be a set. A bounded function $a\colon X \times X \to \mathbf{C}$ is called *positive definite* if, for any $\alpha_i \in \mathbf{C}$ and any finite $F \subset X$,

$$\sum_{i,j \in F} \alpha_i \bar{\alpha}_j a(i, j) \geqslant 0.$$

We denote by $p(X)$ the set of all positive definite bounded functions on $X \times X$.

It was shown by Schur that if a, $b \in p(X)$, then the Hadamard product $a \cdot b \in p(X)$ $((a \cdot b)(i, j) \overset{\text{def}}{=} a(i, j)b(i, j), i, j \in X)$.

From that result follows that the set $\mathscr{L}(l_2(X))$ of all bounded operators on $l_2(X)$ forms a Banach algebra under the Hadamard product.

Let $V_2(X)$ denote the algebra of all multipliers of the Banach algebra $\mathscr{L}(l_2(X))$ under the pointwise multiplication, i.e.

$$V_2(X) = \{a\colon a \cdot \mathscr{L}(l_2(X)) \subset \mathscr{L}(l_2(X))\}.$$

If S and T are two spaces of functions on some set X, let $M(S, T)$ denote the space of all multipliers from S into T, i.e. the space of the functions k on X such that $k \cdot f \in T$ for every $f \in S$. For $M(S, S)$ we write $M(S)$.

A. Grothendieck **[8]** observed that

$$V_2(X) = M(c_0(X) \hat{\otimes} c_0(X)).$$

J. E. Gilbert **[6]** and G. Bennet **[1]** showed that $V_2(X) = \{\langle x(i), y(j)\rangle\colon x(i), y(j) \in \text{Hilbert space and } \|x(i)\| \leqslant C, \|y(j)\| \leqslant C\}$.

From the last theorem follows that $V_2(X)$ is the linear span of the $p(X)$.

The last space was investigated in an excellent way by M. G. Krein **[10]**.

The Littlewood inequality (essentially its dual form) says that if the matrix a defines a continuous linear operator from $l_1(X)$ to $l_2(X)$, then $a \in V_2(X)$, and

$$\|a\|_{V_2} \leqslant \sqrt{2} \sup_{i \in X} \left(\sum_{j \in X} |a(i, j)|^2 \right)^{1/2}.$$

Received by the editors July 16, 1984.

1980 *Mathematics Subject Classification*. Primary 20E05, 22B05, 43A07, 43A15.

Key words and phrases. Fourier-Stieltjes algebra, Herz-Schur multipliers, amenable groups, Littlewood functions.

[1] This work was done during a visiting stay at the University of Heidelberg, June–July 1984. I want to thank Professor M. Leinert for his warm hospitality and a nice working atmosphere.

N. Varopoulos [16] considered the set of Littlewood functions

$$t_2(X) = \{a_1 + a_2 : a_1 \in \mathscr{L}(l_1, l_2(X)), a_2 \in \mathscr{L}(l_2, l_\infty(X))\}$$

and he showed that $a \in t_2(X)$ iff the norm

$$\|a\|_{t_2}^2 = \sup_{F_1, F_2\text{-finite}} \left\{ \frac{1}{|F_1|} \sum |a(i, j)|^2 : i \in F_1, j \in F_2, |F_1| = |F_2| \right\}$$

is finite.

From now on let $X = G$ be a discrete group and let G act on itself by left translation. We call a function a on $X \times X$ invariant if $a(gx, gy) = a(x, y)$ for any $x, y, g \in G$, i.e. there exists a function f on G such that $f(y^{-1}x) = a(x, y)$. Let

$$VN(G) = \mathscr{L}(l_2(G))^{\text{inv}}$$

be the set of all invariant operators on $l_2(G)$—the von Neumann algebra of the group G considered by P. Eymard, which is the dual of the Fourier algebra $A(G) = l_2(G) * l_2(G)$.

Let $B_2(G) = V_2(G)^{\text{inv}}$ denote the Herz-Schur algebra. For another interesting characterization of the Herz-Schur algebra $B_2(G)$ see the paper [3], where it was shown that $B_2(G) = M_0A(G)$, where $M_0A(G)$ is the space of all completely bounded multipliers of the Fourier algebra $A(G)$. For $M_0A(G)$ see also the paper [4] of de Cannière and U. Haagerup.

Let $P(G) = p(G)^{\text{inv}}$ be the set of all positive definite functions on G, $B(G) =$ lin $P(G)$.

Let us note that $B(G)$ is the dual of the full C^*-algebra $C^*(G)$ of the group G.

It was proved by M. G. Krein [10] that if G is an amenable group, then $B(G) = B_2(G)$.

It was shown by C. Herz [9] that, for any locally compact group G,

$$B(G) \subset B_2(G) \subset M(A(G)).$$

For the free group $\mathbf{F}_2$, M. Leinert [11] has observed that $B(\mathbf{F}_2) \subsetneq B_2(\mathbf{F}_2)$; also in the paper [2] it was noted that $B_2(\mathbf{F}_2) \subsetneq M(A(\mathbf{F}_2))$.

C. Nebbia [13] proved that a discrete group G is amenable iff $B(G) = M(A(G))$. V. Losert [18] extended that result to all l.c. groups.

The aim of this note is to replace Nebbia's result by the following stronger statement:

$$B(G) = B_2(G) \quad \text{iff the group } G \text{ is amenable.}$$

Let us now recall that the Banach space W is called of cotype 2 if there exists a constant $C > 0$ such that, for any $x_1, x_2, \dots, x_n \in W$ and any $n = 1, 2, 3, \dots,$ we have

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \geqslant C \left(\sum_{k=1}^n \|x_k\|^2 \right)^{1/2}$$

where r_n are the Rademacher functions on $[0, 1]$. It is well known that L^p-spaces $(1 \leqslant p \leqslant 2)$ are of cotype 2.

2. The Theorem.

THEOREM. *For a discrete group G the following conditions are equivalent*:
(i) *G is an amenable group.*
(ii) $B_2(G) = B(G)$.
(iii) *$B_2(G)$ is of cotype* 2.

PROOF. (i) → (ii) was given by M. G. Krein [**10**]; (ii) → (iii) follows from the N. Tomczak-Jaegermann [**15**] theorem that the dual of a C^*-algebra is of cotype 2. (See also G. Pisier [**14**] for a simple proof of that fact.) We show now that (iii) → (i).

First we show that $M(l_\infty(G), B_2(G)) \subset l_2(G)$. Let $g \in M(l_\infty(G), B_2(G))$ and let $g = \sum_{n=1}^\infty \alpha_n \delta_{x_n}$, then for each $t \in [0,1]$ the function

$$g_t = \sum_{n=1}^\infty \alpha_n r_n(t) \delta_{x_n} \in B_2(G) \quad \text{and} \quad \|g_t\|_{B_2} \leqslant \|g\|_{M(l_\infty, B_2)}.$$

Since by assumption $B_2(G)$ has cotype 2 we get

$$\|g\|_{M(l_\infty, B_2)} \geqslant \int_0^1 \left\| \sum \alpha_n r_n(t) \delta_{x_n} \right\|_{B_2} dt \geqslant C \left(\sum_{n=1}^\infty |\alpha_n|^2 \right)^{1/2}.$$

Hence $g \in l_2(G)$.

Now let us observe that the set of all *Littlewood functions*

$$T_2(G) = t_2^{\mathrm{inv}}(G) \subset M(l_\infty(G), B_2(G)),$$

therefore $T_2(G) \subset l_2(G)$. On the other hand, we always have $l_2(G) \subset T_2(G)$, so $T_2(G) = l_2(G)$.

Let us note (see also J. Wysoczański's general result [**17**]) that from the Varopoulos characterization of the Littlewood functions we have, for $f \in T_2(G)$,

$$\|f\|_{T^2}^2 = \sup_{|F_1| = |F_2| < \infty} \left\{ \frac{1}{|F_1|} \langle |f|^2, \chi_{F_1} * \check{\chi}_{F_2} \rangle \right\} \leqslant \left\| |f|^2 \right\|_{VN(G)}.$$

Since $T_2(G) = l_2(G)$ we get

$$\| |f| \|_{l_2(G)}^2 \leqslant C_1^2 \left\| |f|^2 \right\|_{VN(G)} \quad \text{for every } f \in T_2(G).$$

This implies that for any positive function $g \in l_1(G)$ we have

$$\|g\|_{l_1(G)} \leqslant C \|g\|_{VN(G)}.$$

Hence by the Kesten-Hulanicki characterization of amenable groups we obtain that G is amenable.

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