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THE POTENTIAL $\mathcal{J}$-RELATION AND AMALGAMATION BASES FOR FINITE SEMIGROUPS

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ABSTRACT. Let S be a finite semigroup, $a, b \in S$. When does there exist a finite semigroup T containing S such that $a \, J \, b$ in T? This problem was posed to the second named author by John Rhodes in 1974. We show here that if a, b are regular, then such a semigroup T exists if and only if either $a \, J \, b$ in S, or $a \notin SbS$ and $b \notin SaS$. We use this result to show that amalgamation bases for the class of finite semigroups have linearly ordered $\mathcal{J}$-classes.

1. Preliminaries. For X any set, $|X|$ denotes the cardinality of X and $\mathcal{T}(X)$ denotes the full transformation semigroup on X, acting on the right on X. If $a \in \mathcal{T}(X)$, then $\rho(a)$ denotes the rank of a, namely $|Xa|$. Let F be a field. Then we let $M_n(F)$ denote the multiplicative monoid of all $n \times n$ matrices over F. If $a \in M_n(F)$, then $\rho(a)$ denotes the rank of a. Let X be a set, with $|X| = n$. Then $\mathcal{T}(X)$ acts naturally on a vector space of dimension n over F. Thus $\mathcal{T}(X)$ embeds naturally into $M_n(F)$. Moreover, this embedding preserves rank.

Let S be a semigroup. If $a, b \in S$, then we write $a|b$ (a *divides* b) if $b \in S^1aS^1$, i.e., if $J_b \le J_a$.

By an *amalgam* we mean a list $(S, T; U)$ of semigroups such that $S \cap T = U$, and we say the amalgam is *embeddable* if there is a semigroup W with S and T as subsemigroups (see [**5**] for more details). By an *amalgamation base* for a class $\mathcal{C}$ of semigroups we mean any $U \in \mathcal{C}$ such that every amalgam $(S, T; U)$ with $S, T \in \mathcal{C}$ is embeddable in some $W \in \mathcal{C}$ (see [**4**] for some examples).

2. The potential $\mathcal{J}$-relation.

THEOREM 1. *Let S be a finite semigroup, $A \subseteq S$.*

(a) *If for any $a, b \in A$, either $a \, J \, b$ or $a \nmid b$, $b \nmid a$, then there exists a finite semigroup T containing S such that A lies in a $\mathcal{J}$-class of T. If S is an inverse semigroup, then T can be chosen to be an inverse semigroup.*

(b) *Conversely, if S can be embedded in a finite semigroup T containing A within a $\mathcal{J}$-class and if every element of A is regular in S, then for any $a, b \in A$, either $a \, J \, b$ in S or $a \nmid b$, $b \nmid a$ in S.*

PROOF. (a) Without loss of generality we can assume that $S = S^1$ and that A contains no pair of $\mathcal{J}$-equivalent elements. We shall embed S in a transformation semigroup $\mathcal{T}(X)$ such that the elements of A all have the same rank, i.e., are $\mathcal{J}$-related.

Choose $a \in A$ such that $|Sa|$ is the maximum. Let $A = \{a = a_0, a_1, \ldots, a_p\}$. Let $I_j = \bigcup_{k \ne j} Sa_kS$, $j = 1, \ldots, p$. Let $S_0 = S$, $S_j = S/I_j$, $j = 1, \ldots, p$. Then S acts on the right on S_i, $i = 0, \ldots, p$. Let $|Sa_i| = m_i$, $i = 0, \ldots, p$, $\beta_j = |S_j a_j| > 1$,

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$j = 1, \ldots, p$, $\alpha_0 = \prod_{j=1}^{p}(\beta_j - 1)$ and $\alpha_j = \alpha_0(m_0 - m_j)/(\beta_j - 1)$, $j = 1, \ldots, p$. So $\alpha_0 \geq 1$, $\alpha_j \geq 0$, $j = 1, \ldots, p$. Let X denote the disjoint union of α_i copies of S_i, $i = 0, \ldots, p$. Then S acts faithfully on the right on X and thus S embeds in $\mathcal{T}(X)$. Clearly $|Xa_0| = \alpha_0 m_0 + \sum_{j=1}^{p} \alpha_j$, and $|Xa_j| = \alpha_0 m_j + \alpha_j \beta_j + \sum_{k \neq 0, j} \alpha_k$, $j = 1, \ldots, p$. Routine calculations now show that $|Xa_0| = |Xa_j|$, $j = 1, \ldots, p$. So A lies within a $\mathcal{J}$-class of $\mathcal{T}(X)$.

If S is an inverse semigroup, then we can use the Preston-Vagner representation of S and the S_j's to obtain a representation of S in the symmetric inverse semigroup on X.

(b) Suppose that S can be so embedded in T and suppose, to the contrary, that there exist $a, b \in A$ such that $a|b$ and $b \nmid a$. By [**2**, Theorem 1 or **7**, Proposition 3.1] there exist idempotents $e \in J_a$, $f \in J_b$ such that $e > f$. Clearly $e \mathcal{J} f$ in T, so T contains a copy of the bicyclic semigroup [**1**, Theorem 2.54] and hence is infinite, a contradiction.

The next theorem shows that with respect to semigroup division, the whole semigroup can be put into a $\mathcal{J}$-class.

THEOREM 2. *Let S be a finite semigroup. Then there exists a finite regular semigroup T, a subsemigroup T_0 of T, a $\mathcal{J}$-class J of T and a morphism $\phi: T_0 \to S$ such that $\phi(J \cap T_0) = S$. If S is a regular semigroup, then T_0 can be chosen to be a regular semigroup. If S is an inverse semigroup, then T_0 and T can be chosen to be inverse semigroups.*

PROOF. Let $T_0 = S \times \{0, 1\}$ with the following multiplication:

$$(a, \alpha)(b, \beta) = (ab, \gamma),$$

where

$$\gamma = \begin{cases} 1 & \text{if } \alpha = \beta = 1 \text{ and } a \mathcal{J} b \mathcal{J} ab, \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that T_0 is a semigroup. If S is a regular semigroup, then T_0 is a regular semigroup. If S is an inverse semigroup, then T_0 is an inverse semigroup. The map $a \mapsto (a, 0)$ embeds S into T_0.

Easy manipulation (or Remark 1 below) now shows that the subset $A = S \times \{1\}$ satisfies the hypothesis of Theorem 1(a). Thus there exists a finite semigroup T containing T_0 such that A lies in a $\mathcal{J}$-class J of T. the map $\phi: T_0 \to S$ given by $\phi(a, \alpha) = a$ is a morphism and $\phi(J \cap T_0) \supseteq \phi(A) = S$. This proves the theorem.

REMARK 1. In fact $S \times \{0\}$ is an ideal of T_0 and $T_0/(S \times \{0\})$ is isomorphic to the 0-direct union [**1**, §6.3] of all the principal factors $\{J \cup \{0\}: J \in S/\mathcal{J}\}$ (including, if $S = S^0$, the two element semilattice), while T_0 is the ideal extension of $S \times \{0\}$ (or S) by $T_0/(S \times \{0\})$ determined by the partial morphism $(s, 1) \mapsto (s, 0)$ for $s \in S$.

3. Amalgamation bases.

THEOREM 3. *For any amalgamation base of the class of finite [finite regular, finite inverse] semigroups, the $\mathcal{J}$-classes are linearly ordered.*

PROOF. Take any amalgamation base, U say, of the class of finite semigroups, and suppose, to the contrary, that there are two elements, a and b say, whose $\mathcal{J}$-classes are not comparable.

Case I. a or b is regular, say a. Take any idempotent $e \in J_a$. Form a semigroup $U' = U \cup \{e'\}$ containing U as a subsemigroup (where $e' \notin U$) by defining $e'^2 =$

e', $e'u = eu$, $ue' = ue$, for all $u \in U$. Then $e' \nmid b$ and $b \nmid e'$ in U', so by Theorem 1 there exists a finite semigroup S containing U' such that $c' \mathcal{J} b$ in S.

Since $J_e = J_a$ and J_b are not comparable, again by Theorem 1, there exists a finite semigroup T containing U such that $e \mathcal{J} b$ in T.

The amalgam $(S, T; U)$ is embeddable in a finite semigroup, W say, since U is an amalgamation base for the class of finite semigroups. But then in W we have $e \mathcal{J} b \mathcal{J} e'$ and $e < e'$, whence W contains a copy of the bicyclic semigroup [**1**, Theorem 2.54] and is infinite, a contradiction.

The proof so far is easily modified to give a proof of the bracketed statements. Note that the result for the class of finite regular semigroups is a trivial corollary of the result for the class of finite semigroups, since a regular semigroup is an amalgamation base for either class if and only if it is one for the other class (since any finite semigroup embeds in a finite regular semigroup).

Case II. a and b are not regular. Without loss of generality we can assume that U is a subsemigroup of $M_n(\mathbf{Z}_2)$ for some positive integer n, and that $n > \rho(a) \geq \rho(b)$. Consider the embedding $\theta\colon M_n(\mathbf{Z}_2) \to M_{2n}(\mathbf{Z}_2)$ given by $\theta(c) = \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$. Put $a' = \left(\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}\right)$, where 1 denotes the $n \times n$ identity matrix; then $\rho(a') = n + \rho(a) > 2\rho(a) = \rho(\theta(a)) \geq \rho(\theta(b))$ and $a'^2 = \theta(a)$. (This method of finding square roots is a variation of that due to C. J. Ash [**5**, Theorem 5.1].) Thus, so far, we have embedded U in a finite semigroup U' with an element a' such that $a'^2 = a$ and $b \nmid a'$ (note that from $b \nmid a$ in U we do not get $b \nmid a$ in U', so it is not immediate from $a'^2 = a$ that $b \nmid a'$ in U').

Now consider the embedding $\psi\colon U \to U' \times (U/U^1aU^1)$ given by $\psi(u) = (u, \phi(u))$, where ϕ is the canonical morphism of U upon U/U^1aU^1. Put $v = (a', 0)$; then $v^2 = (a, 0) = \psi(a)$ and $v \nmid (b, \phi(b)) = \psi(b)$ since $\phi(b) \neq 0$. Also $\psi(b) \nmid v$ since $b \nmid a'$ in U'. Thus we have a semigroup V containing U and an element v such that $v^2 = a, v \nmid b, b \nmid v$.

By Theorem 1, there exists a finite semigroup S containing V such that $v \mathcal{J} b$ in S. Also, since in U, J_a and J_b are not comparable, by Theorem 1 there is a finite semigroup T containing U such that $a \mathcal{J} b$ in T.

Since U is an amalgamation base for the class of finite semigroups, the amalgam $(S, T; U)$ is embeddable in a finite semigroup W, say. Then $v \mathcal{J} b \mathcal{J} a = v^2$ in W, and since W is finite, we have $v \mathcal{H} v^2 = a$ in W. Thus a is in a subgroup of W and hence in a subgroup of U, contradicting that a is not regular in U.

REMARK 2. The existence of a finite inverse semigroup which is not an amalgamation base for the class of finite inverse semigroups was first shown by C. J. Ash: his example, given in [**3**], is the three element semilattice which is not a chain. His construction and proof led us to the proof in Case I above.

REMARK 3. One of the authors has recently shown that the $\mathcal{J}$-classes being linearly ordered is also a sufficient condition for a finite inverse semigroup to be an amalgamation base of the class of finite inverse semigroups.

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