EXPONENTIAL SUMS AND FORMS FOR VARIETIES OVER FINITE FIELDS

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ABSTRACT. We prove that the roots of the L -function of an Artin-Schreier cover of an algebraic variety defined over a finite field differ from the roots of the zeta function of the cover by roots of unity.

In this paper we use an observation about forms for varieties to prove that the roots of the L -function of an Artin-Schreier cover of an algebraic variety defined over a finite field differ from the roots of the zeta function of the cover by roots of unity. This answers affirmatively a conjecture of Bombieri [1].

More specifically, let X be an absolutely irreducible normal variety defined over the finite field $k = \mathbb{F}_q$ with $q = p^a$ elements. Let $\bar{k}$ be an algebraic closure of k , and $k(X)$, $\bar{k}(X)$ the function field of X over k and $\bar{k}$, respectively. Let $R(x) \in k(X)$ be such that $R(x) \neq h^p - h$ for $h \in \bar{k}(X)$. Let X_m be the set of points of X defined over $\mathbb{F}_{q^m}$, and let

$$S_m(R, X) = \sum' \exp[2\pi i \operatorname{tr}(R(x))/p]$$

where the sum is over $x \in X_m$, the $'$ means exclude the points lying on the poles of $R(x)$, and $\operatorname{tr}: \mathbb{F}_q \rightarrow \mathbb{F}_p$ is the absolute trace.

Let Y be the variety defined by $Z^p - Z = R(x)$. Then there exists a Galois covering $\pi: Y \rightarrow X$ sending $(z, x) \rightarrow x$. This covering has Galois group $\mathbb{F}_p$ acting on Y by $(z, x) \rightarrow (z + g, x)$, $g \in \mathbb{F}_p$. π is called an *Artin-Schreier covering* of X related to $R(x)$. Define the L -function of this covering by

$$L(t, R, X) = \exp\left(\sum_{m=1}^{\infty} S_m(R, X) t^m/m\right).$$

For further information see the paper of Bombieri [1].

Let $Z(t, Y, q)$ denote the zeta function of Y relative to the field of definition k . Bombieri [1, p. 105] conjectured that if θ is a characteristic root, i.e. a zero or pole of $L(t, R, X)$, then there exists a characteristic root ω of $Z(t, Y, q)$ and a p^u th root of unity γ for some integer u such that $\theta = \gamma\omega$. In this paper we prove this conjecture.

We first recall a theorem of Bombieri [1, Theorem 7]. Let $Y(\lambda)$ be the Artin-Schreier cover associated to $Z^p - Z = R(x) + \lambda$ where $\lambda \in \mathbb{F}_{q^p}$.

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THEOREM 1 (BOMBIERI [1]). *$L(t, R, X)$ is a rational function of t . There exists a suitable power p^u of the degree p of the cover $\pi: Y \rightarrow X$ such that, if θ is a characteristic root of $L(t, R, X)$, then there exists $v \leq u - 1$ and $\lambda \in \mathbb{F}_{q^{p^v}}$ such that $\theta^{p^u} = w^{p^{u-v}}$ where w is a characteristic root of $Z(t, Y(\lambda), q^{p^v})$.*

Let k' be the extension of k of degree s .

DEFINITION 1. Let V be an algebraic variety over k . A k'/k -form for V is a pair (V', f) where V' is an algebraic variety over k and $f: V' \rightarrow V$ is an isomorphism defined over k' .

It is easy to see that, if (V', f) is a k'/k -form for V , then the characteristic roots of $Z(t, V', q)$ differ from those of $Z(t, V, q)$ by s th roots of unity.

LEMMA 1. *If w is a characteristic root of $Z(t, Y(\lambda), q^{p^v})$ and $\lambda \in \mathbb{F}_{q^{p^v}}$ is of the form $c^p - c$ for some $c \in \mathbb{F}_{q^{p^{v+1}}}$, then w^p is a characteristic root of $Z(t, Y, q^{p^{v+1}})$.*

PROOF. $Z^p - Z = R(x) + \lambda = R(x) + c^p - c$, i.e. $(Z - c)^p - (Z - c) = R(x)$. Thus, $Y(\lambda)$ and Y are $\mathbb{F}_{q^{p^{v+1}}}/\mathbb{F}_{q^{p^v}}$ -forms. Therefore, ζw is a characteristic root of $Z(t, Y, q^{p^v})$ where ζ is some p th root of unity. Therefore w^p is a characteristic root of $Z(t, Y, q^{p^{v+1}})$. This proves the lemma.

Now observe that $(w^p)^{p^{v-1}} = w^{p^{p^v}}$ is a characteristic root of $Z(t^{p^{v+1}}, Y, q^{p^{v+1}})$. But

$$Z(t^{p^{v+1}}, Y, q^{p^{v+1}}) = \prod_{\xi} Z(\xi t, Y, q)$$

where the product is over all p^{v+1} th roots of unity. Therefore we must have that $w^{p^{p^v}}$ is a characteristic root of one of the factors $Z(\xi t, Y, q)$. This proves the following lemma.

LEMMA 2. *If w is a characteristic root of $Z(t, Y(\lambda), q^{p^v})$, then $w^{p^{p^v}} = \zeta r$ where r is a characteristic root of $Z(t, Y, q)$ and ζ is a p^{v+1} th root of unity.*

Now, by Theorem 1, we have that if θ is a characteristic root of $L(t, R, X)$, then

$$\theta^{p^u} = w^{p^{u-v}} = (w^{p^{p^v}})^{p^u}, \quad v \leq u - 1,$$

where w is a characteristic root of $Z(t, Y(\lambda), q^{p^v})$ for some $\lambda \in \mathbb{F}_{q^{p^v}}$ of the form $c^p - c$, $c \in \mathbb{F}_{q^{p^{v+1}}}$. Taking p^u th roots we get

$$\theta = \eta w^{p^{p^v}} = (\eta \zeta) r$$

where η is a p^u th root of unity, ζ is a p^{v+1} th root of unity and r is a characteristic root of $Z(t, Y, q)$. Since $v + 1 \leq u$, $\eta \zeta$ is a p^u th root of unity. Letting $\gamma = \eta \zeta$, $\omega = r$ we have the following theorem.

THEOREM 2. *If θ is a characteristic root of the L -function $L(t, R, X)$ associated to the Artin-Schreier cover $Y: Z^p - Z = R(x)$ of X , then $\theta = \gamma \omega$ where γ is a p^u th root of unity and ω is a characteristic root of $Z(t, Y, q)$.*

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