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A SMALL BOUNDARY FOR H^∞ ON A STRICTLY PSEUDOCONVEX DOMAIN

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ABSTRACT. Let $n \geqslant 2$ and $D \subset\subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^k boundary for $k > 2$. There is a closed nowhere dense subset of the maximal ideal space of $L^\infty(\mathrm{b}D)$ which defines a closed boundary for $H^\infty(D)$.

Let $D \subset\subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^k boundary $k > 2$, denoted by bD. By taking nontangential limits, each $f \in H^\infty(D)$ defines almost everywhere on bD, with respect to the induced Lebesgue measure m on bD, a function $f^* \in L^\infty(\mathrm{b}D)$ [**10**]. Let $M(L^\infty(\mathrm{b}D))$ and $M(H^\infty(D))$ be the maximal ideal spaces of $L^\infty(\mathrm{b}D)$ and $H^\infty(D)$, respectively. A closed boundary for $H^\infty(D)$ is a closed subset β of $M(H^\infty(D))$ such that $\|f\| = \sup\{|\phi(f)| \text{ for } \phi \in \beta\}$ for all $f \in H^\infty(D)$. It is possible to define a continuous map π: $M(L^\infty(\mathrm{b}D)) \to M(H^\infty(D))$ by $\pi(\phi)(f) = \phi(f^*)$ for $\phi \in M(L^\infty(\mathrm{b}D))$ and $f \in H^\infty(D)$.

The problem to find information about the Shilov boundary, i.e., the smallest closed boundary, $S(H^\infty(D))$ of $H^\infty(D)$ has been solved in the case of the unit disc because the map π is a homeomorphism from $M(L^\infty(\mathrm{b}\Delta))$ onto $S(H^\infty(\Delta))$ [**6**]. Range has shown that the corresponding result is false, for $n \geqslant 2$, in the case of the polydisc [**8**] and the unit ball.

The main theorem in this paper shows that if D is a strictly pseudoconvex domain, it is indeed possible to construct a closed nowhere dense subset of $M(L^\infty(\mathrm{b}D))$ which defines a closed boundary for $H^\infty(D)$.

The first step involves a local parametrization of the boundary of the domain by boundaries of planar regions, in analogy to the construction in [**8**]. This parametrization is more explicit than the one by analytic discs introduced by Bishop [**2**] and it seems more appropriate for the applications considered. In order to study the properties of the map π we need to use inner functions, whose existence follows from the result of Aleksandrov [**1**] and its generalization to strictly pseudoconvex domains by Løw [**7**].

The transition from the local situation to the general one will be done by using the localization principle for the maximal ideal space of $H^\infty(D)$ obtained by Range [**9**]; the result is stated as Theorem 5. In the present paper we discuss the proof of the main results. The proofs of several technical lemmas are omitted. Full details are given in [**3**].

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Let $D \subset\subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^k boundary ($k > 2$, $n \geqslant 2$). If $q \in \mathrm{b}D$, without loss of generality, after a suitable holomorphic change of coordinates, it is possible to assume that $q = (0,\ldots,0)$ and that in a small enough neighborhood of the origin, called B, D has defining function

$$\gamma(z_1,\ldots,z_n) = -x_1 + \psi(x_2,\ldots,x_n, y_1,\ldots,y_n)$$
$$= -x_1 + a_1 y_1^2 + \sum_{j=2}^{n} a_j|z_j|^2 + \sum_{j=2}^{n} b_j x_j y_1 + \sum_{j=2}^{n} c_j y_j y_1 + O(3).$$

For $p \in \mathrm{b}D$, B with $p = (\alpha_1 + i\beta_1,\ldots,\alpha_n + i\beta_n) = (p_1,\ldots,p_n)$ and $\nu \in N^* = \{1,2,3,\ldots\}$, define

$$C_p = \{ \lambda \in B | \lambda_j = p_j \text{ for } j = 1,\ldots,n-1 \text{ and } \lambda_n = x_n + iy_n$$
$$\text{such that } \alpha_1 = \psi(\alpha_2,\ldots,\alpha_{n-1}, x_n, \beta_1,\ldots,\beta_{n-1}, y_n)\} \subset \mathrm{b}D,$$
$$\Delta_p = \{ \lambda \in B | \lambda_j = p_j \text{ for } j = 1,\ldots,n-1 \text{ and } \lambda_n = x_n + iy_n$$
$$\text{such that } \alpha_1 > \psi(\alpha_2,\ldots,\alpha_{n-1}, x_n, \beta_1,\ldots,\beta_{n-1}, y_n)\} \subset D,$$
$$\Delta_{p,\nu} = \{ \lambda \in B | \lambda_j = p_j \text{ for } j = 1,\ldots,n-1 \text{ and } \lambda_n = x_n + iy_n$$
$$\text{such that } \alpha_1 > \psi(\alpha_2,\ldots,\alpha_{n-1}, x_n, \beta_1,\ldots,\beta_{n-1}, y_n) + 1/\nu\} \subset \Delta_p.$$

In the local representation, D is convex. Δ_p is the intersection of it with a $(2n-2)$-dimensional plane, so Δ_p is itself convex. In particular, it is a connected domain in a complex plane. Then its boundary C_p is a simple closed curve.

LEMMA 1. *Consider the singularity set $S = \{z \in \mathrm{b}D \cap B | z \in C_p$ for some $p \in \mathrm{b}D \cap B$ for which there exists a point $\hat{p} \in C_p$ such that $(\partial\gamma/\partial x_n)(\hat{p}) = 0$ and $(\partial\gamma/\partial y_n)(\hat{p}) = 0\}$. Then $m(S) = 0$.*

PROOF. One shows that S has measure zero locally by applying the Implicit Function Theorem to prove that S is stratified by real submanifolds of $\mathrm{b}D \cap B$ of dimension less than $2n - 1$. (To do so the strict convexity of the domain is needed.) Define

$$S_1 = \left\{ z \in B | \gamma(z) = 0;\ \frac{\partial\gamma}{\partial x_n}(z) = 0;\ \frac{\partial\gamma}{\partial y_n}(z) = 0 \right\} \quad \text{and} \quad S_2 = S - S_1.$$

Since $\det J(\gamma, \partial\gamma/\partial x_n, \partial\gamma/\partial y_n)(0,\ldots,0) \neq 0$ by continuity and by the Implicit Function Theorem, it follows that in a small enough neighborhood of the origin S_1 is a real submanifold of dimension $2n - 3$. Then $m(S_1) = 0$. Let $p \in S_2$. We can assume, without loss of generality that $(\partial\gamma/\partial x_n)(p) \neq 0$. Then in a neighborhood $V(p)$ of the point p, by the Implicit Function Theorem, it is possible to write $x_n = h(x_1,\ldots,x_{n-1}, y_1,\ldots,y_n)$ with $h \in C^1$. By definition of S_2, there exists at least a $q \in S_1 \cap C_p$. It is possible to apply again the Implicit Function Theorem to get that, in a neighborhood of q, $B(q, \varepsilon(q))$, S_1 can be described by $(x_1, x_n, y_n) = k^q(x_2,\ldots,x_{n-1}, y_1,\ldots,y_{n-1})$, where k^q is at least of class C^1. The points of $S_1 \cap B(q, \varepsilon(q))$ identify a subset Γ_q of S_2 in a neighborhood $U_q(p)$ of p with $U_q(p) \subset V(p)$ where

$$\Gamma_q = \{ z \in U_q(p) | x_1 = k_1^q(x_2,\ldots,y_{n-1});\ x_n = h(x_1,\ldots,x_{n-1}, y_1,\ldots,y_n)\}.$$

Γ_q has measure zero. By a compactness argument, it is possible to prove that there exists an open neighborhood of p, V, such that

$$V \cap S_2 \subset \bigcup_{\substack{j=1 \\ q_j \in C_p \cap S_1}} \Gamma_{q_j}.$$

Then S_2 has local measure zero; so $m(S_2) = 0$.

Parametrization of a neighborhood of a nonsingular point. As a consequence of Lemma 1, the set $(\mathrm{b}D \cap B) - S$ is dense and open in $\mathrm{b}D \cap B$. Fix $p_0 \in (\mathrm{b}D \cap B) - S$, with $p_0 = (x_1^0 + iy_1^0, \ldots, x_n^0 + iy_n^0)$. Then there exists a neighborhood of p_0, $B_1(p_0)$, contained in $(\mathrm{b}D \cap B) - S$. If $p \in B_1(p_0)$, for all $\zeta \in C_p$ one has that either $(\partial\gamma/\partial x_n)(\zeta) \neq 0$ or $(\partial\gamma/\partial y_n)(\zeta) \neq 0$. Then C_p is a differentiable curve in the plane $\{z \in C^n | z_j = p_j \text{ for } j < n\}$ and it can be parametrized by using the arc length. The choice of the initial point $\hat{p} \in C_p$, for the parametrization of C_p, can be made to depend on p_0 in a differentiable way, using again the Implicit Function Theorem in a neighborhood of p_0. Let s_p be the parametrization of C_p by arc length, normalized so that the total length of C_p is one and so that $s_p(0) = \hat{p}$. Define then the function S_p: $[0,1] \to \mathrm{b}D \cap B$ given by $S_p(t) = z = (p_1, \ldots, p_{n-1}, z_n(t))$ where $z_n(t)$ is the point having normalized distance t from $\hat{p}$ on C_p. It is possible to extend this correspondence to all real numbers and points in C_p by setting $z_n(t \pm 1) = z_n(t)$ for $t \in \mathbf{R}$. Define then β_p: $S^1 \to C_p$ as $\beta_p(e^{it}) = (p_1, \ldots, p_{n-1}, z_n(t/2\pi))$. β_p is one-to-one and onto.

For a suitable $\varepsilon_0 = \varepsilon(p_0)$, it is possible to assume that the open neighborhood of p_0 in which the construction can be carried out is of the kind

$$U(p_0) = \left\{ z \in \mathrm{b}D \big| \left|x_j - x_j^0\right| < \varepsilon_0;\ \left|y_j - y_j^0\right| < \varepsilon_0 \text{ for } j < n;\ \left\|z_n - z_n^0\right\| < \varepsilon_0 \right\}$$

with $\overline{U(p_0)} \subset (\mathrm{b}D \cap B) - S$. Then if $z \in U(p_0)$, it follows that C_z can be parametrized by S_z, which depends differentiably on z, as done before. Set

$$I_{0,j} = \left[x_j^0 - \varepsilon_0, x_j^0 + \varepsilon_0\right] \quad \text{and} \quad J_{0,j} = \left[y_j^0 - \varepsilon_0, y_j^0 + \varepsilon_0\right],$$

$$K_{0,j} = \left\{ \zeta \in C | x \in I_{0,j} \text{ and } y \in J_{0,j} \right\},$$

$$Q_0 = \prod_{j<n} K_{0,j} \times S^1 = \left(\prod_{j<n} I_{0,j} \times J_{0,j} \right) \times S^1.$$

Consider $\tilde{\Phi}_0$: $Q_0 \to \mathrm{b}D$ given by

$$\tilde{\Phi}_0\left(z_1, \ldots, z_{n-1}, e^{it}\right) = \left(z_1, \ldots, z_{n-1}, z_n(t/2\pi)\right).$$

If $\Omega_0 = \tilde{\Phi}_0(Q_0)$, define Φ_0: $Q_0 \to \Omega_0$ given by

$$\phi_0\left(z_1, \ldots, z_{n-1}, e^{it}\right) = \tilde{\Phi}_0\left(z_1, \ldots, z_{n-1}, e^{it}\right).$$

LEMMA 2. *Φ_0 is a diffeomorphism from the interior of Q_0 onto $\Omega_0 - \Phi_0(\mathrm{b}Q_0)$.*

PROOF. It follows from the construction that the jacobian of Φ_0 is different from zero.

Since $(\mathrm{b}D \cap B) - S$ is an open dense set in $\mathrm{b}D \cap B$, it is possible to cover it with a countable collection of sets obtained by repeating the construction of Ω_0.

Local construction of a lower semicontinuous function corresponding to an $H^\infty(D)$ function. Let $f \in H^\infty(D)$ and $p_0 \in (\mathrm{b}D \cap B) - S$. Then p_0 has a parametrizable neighborhood Ω_0. Consider $W = \bigcup_{\lambda \in \mathrm{Int}\,\Omega_0} \Delta_\lambda$ where the interior Ω_0 is taken with respect to $\mathrm{b}D$. For $\lambda \in \Omega_0$ define $G_f(\lambda) = \sup_{\Delta_\lambda} |f(z)|$. Let

(1) $\tilde{f}^*$ be the nontangential limit of f taken on Δ_λ. By Fatou's Theorem this limit exists a.e. on $\mathrm{b}\Delta_\lambda = C_\lambda$ with respect to arc length on $\mathrm{b}\Delta_\lambda$,

(2) for $\nu \in N^*$, $g_\nu(\lambda) = \sup_{\Delta_{\lambda,\nu}} |f(\zeta)|$.

LEMMA 3. *g_ν is a continuous function for any sufficiently large ν.*

LEMMA 4. *G_f is lower semicontinuous and* $\sup_{\Omega_0} G_f(\lambda) = \|f\|_{H^\infty(W)}$.

PROOF. It follows from Lemma 3 and from the following equalities

$$G_f(\lambda) = \sup_{\Delta_\lambda} |f(z)| = \operatorname*{ess\,sup}_{C_\lambda} |\tilde{f}^*(\zeta)|$$
$$= \lim_{\nu \to \infty} \sup_{\Delta_{\lambda,\nu}} |f(z)| = \sup_{\nu \to \infty} g_\nu(\lambda).$$

The local result can be stated as follows:

THEOREM 1. *There exists a nowhere dense subset of the maximal ideal space of $L^\infty(\Omega_0)$ which defines a closed boundary for $H^\infty(W)$ by using the canonical map π: $M(L^\infty(\Omega_0)) \to M(H^\infty(W))$ induced by the inclusion map between $H^\infty(W)$ and $L^\infty(\Omega_0)$.*

In order to prove it, one needs some additional properties of W and f:

PROPOSITION 1. *W is an open set and* $\Omega_0 = \mathrm{b}W \cap \mathrm{b}D$.

PROOF. The proof involves a straightforward verification.

Let f^* be the nontangential limit of f taken in D. It exists m a.e. by Fatou's Theorem.

THEOREM 2. *If $f \in H^\infty(D)$, then* $\|f\|_{H^\infty(W)} = \|f^*\|_{L^\infty(\Omega_0)}$.

PROOF. Set $a = \|f\|_{H^\infty(W)}$ and $b = \|f^*\|_{L^\infty(\Omega_0)}$.

(i) $a \leqslant b$. Suppose not. Then $|f(z_0)| > b$ for some $z_0 \in W$. Because f is continuous, this inequality holds in a neighborhood $V(z_0)$ of the point. Let $U(z_0) = V(z_0) \cap W$. It is open, nonempty, and in it $|f(\zeta)| > b$. Consider the set $F = \{\lambda \in \mathrm{Int}\,\Omega_0 | \Delta_\lambda \cap U(z_0) \neq \varnothing\}$. It is possible to show that it is open in Ω_0. Then if $w_0 \in F$, it follows that $C_{w_0} \subset F$, and one has that $F \supset \Phi_0(A)$ where A is an open set in Q_0 with $A = A_1 \times S^1$. f^* exists m a.e. on $\bigcup_{\lambda \in F} C_\lambda$. Then by Fubini's Theorem it exists on C_λ a.e. with respect to the measure of it as a curve in a plane for all λ except those in a set of $2n-2$ measure zero. Then there is a set $Q \subset A$ with $m_{2(n-1)}(Q) = 0$ such that f^* and $\tilde{f}^*$ exists a.e. with respect to the m_1 meausre on S^1, at $\tilde{F} = \Phi_0((A_1 \times Q) \times S^1)$. For $\lambda \in \tilde{F}$, $f^* = \tilde{f}^*$ and they exist a.e. on $C_\lambda = \mathrm{b}\Delta_\lambda$. Since $\Delta_\lambda \cap U(z_0) \neq \varnothing$, there exists a $\zeta_\lambda \in \Delta_\lambda$ such that

$$|f(\zeta_\lambda)| > b \geqslant \sup_{\lambda \in \tilde{F}} \|f^*\|_{L^\infty(\mathrm{b}\Delta_\lambda)} \geqslant \|\tilde{f}^*\|_{L^\infty H(\mathrm{b}\Delta_\lambda)}$$

for all $\lambda \in \tilde{F}$. This contradicts the maximum modulus principle.

The construction of G_f can be done, keeping all the above properties, for any $f \in H^\infty(W)$.

PROOF OF THEOREM 1. Ω_0 has been parametrized using $\Phi_0: Q_0 \to \Omega_0$. Let $\{r_l\}_{l=1}^\infty$ be an enumeration of points with rational coordinates in $\prod_{j<n} K_{j,0}$ with $r_l = (x_l^1 + iy_l^1, \ldots, x_l^{n-1} + iy_l^{n-1})$. Let k, m be positive integers and define

(1)

$$I_{k,m}^{(l)} = \left\{ s \in \prod_{j<n} K_{0,j} \middle| \left|x_j - x_l^j\right| < \frac{1}{km}\frac{1}{2^{l+1}};\ \left|y_j - y_l^j\right| < \frac{1}{km}\frac{1}{2^{l+1}} \right\},$$

with $l = 1, \ldots, \infty$,

(2) $I_{k,m} = \bigcup_{l=1}^\infty I_{k,m}^{(l)}$,

(3) $E_{k,m} = \Phi_0(I_{k,m} \times S^1)$.

$E_{k,m}$ is open and dense in Ω_0 and $m(E_{k,m}) \leqslant \tilde{c}/k^2m^2$. Moreover, if $w \in E_{k,m}$, it follows that

$$C_w = \Phi_0\left\{ \left(w_1, \ldots, w_{n-1}, e^{it}\right) \middle| t \in \mathbf{R} \right\} \subset E_{k,m}.$$

Define

$$U_{k,m} = \left\{ \psi \in M(L^\infty(\Omega_0)) \middle| \psi(\chi_{E_{k,m}}) = 1 \right\}.$$

This set is closed and open in [5] and $\hat{m}(U_{k,m}) = (E_{k,m}) < \tilde{c}(km)^{-2}$, where m is the induced measure on $\Omega_0 \subset \mathrm{b}D$ and $\hat{m}$ is the regular Borel measure on $M(L^\infty(\Omega_0))$ such that $\int f\,dm = \int \hat{f}\,d\hat{m}$ for all $f \in L^\infty(\Omega_0)$ [5]. The next step is to prove that $\tau(U_{k,m}) \subset M(H^\infty(W))$ is a boundary for $H^\infty(W)$ where τ is defined by $\tau(\psi)(f) = \psi(f^*)$ for all $f \in H^\infty(W)$. $\tau(U_{k,m})$ is closed because it is compact. For $f \in H^\infty(W)$ define G_f as before with $G_f: \Omega_0 \to \mathbf{R}$.

Assume that $1 \geqslant \sup_{\tau(U_{k,m})} |\hat{f}| = \operatorname{ess\,sup}_{E_{k,m}} |f^*|$ and consider the function $g = |f^*| \circ \Phi_0$. Then $g \in L^\infty(Q_0, m_{2n-1})$ and $g \leqslant 1\ m_{2n-1}$ a.e. on $I_{k,m} \times S^1$. The set

$$F_{k,m} = \left\{ (s,t) \in I_{k,m} \times S^1 \middle| f^*\left(\Phi_0\left(z_1, \ldots, z_{n-1}e^{it}\right)\right) \text{ does not exist} \right\}$$

has measure zero in $I_{k,m} \times S^1$. Then $g \leqslant 1\ m_{2n-1}$ a.e. on $(I_{k,m} \times S^1) - F_{k,m}$. This means that for almost all $(z_1, \ldots, z_{n-1}) \in I_{k,m}$, one has

$$\operatorname*{ess\,sup}_{S^1} g\left(z_1, \ldots, z_{n-1}, e^{it}\right) \leqslant 1$$

that gives $G_f(w) \leqslant 1\ m$ a.e. on $E_{k,m} - \Phi_0(F_{k,m})$, i.e. $G_f(w) \leqslant 1\ m$ a.e. on $E_{k,m}$. Since G_f is lower semicontinuous and $E_{k,m}$ is open and dense in Ω_0, it follows that $G_f(w) \leqslant 1$ for all $w \in \Omega_0$. Therefore, by Lemma 4, it follows that $\|f\|_{H^\infty(W)} \leqslant 1$. This proves that $\tau(U_{k,m})$ is a boundary for $H^\infty(W)$.

Construction of the boundary with the required properties. The sequence $\{U_{k,m}\}$ is a nested sequence because $E_{k,m} \supset E_{k+1,m}$ for all m. Using a compactness argument, the set $\beta = \bigcap_{k=1}^\infty U_{k,m}$ is nonempty. Since $\tau(\beta) = \bigcap \tau(U_{k,m})$ it follows that $\tau(\beta)$ is a closed boundary for $H^\infty(W)$. Moreover, $\hat{m}(\beta) = \lim_{k\to\infty} \hat{m}(U_{k,m}) = 0$. This implies that β is nowhere dense.

THEOREM 3. *The map* $\tau: M(L^\infty(\Omega_0)) \to M(H^\infty(W))$ *is not one-to-one.*

PROOF. The argument given in [8] can be carried over to the situation considered here.

THEOREM 4. *$\tau(M(L^\infty(\Omega_0)))$ is strictly larger than the Shilov boundary for $H^\infty(W)$.*

PROOF. Fix any index k and construct the set $E = E_{k,m}$. It is open, dense in Ω_0, and $m(E) \leqslant \tilde{c}(km)^{-2}$. Then there exists a set $F \subset \Omega_0 - E$ which is closed and of positive measure. Consider $\tilde{f}\colon \Omega_0 \to \mathbf{R}$ to be equal to 1 on F and equal to 2 on $\Omega_0 - F$. It is lower semicontinuous and $\tilde{f} \in L^\infty(\Omega_0)$. It can be trivially extended to an $\tilde{\tilde{f}} \in L^\infty(\mathrm{b}D)$, which is still positive. Then by [7] there exists an $f \in H^\infty(D)$ such that $f^* = \tilde{\tilde{f}}\, m$ a.e. on $\mathrm{b}D$. Then $f/W \in H^\infty(W)$ and $(f/W)^* = \tilde{f}\, m$ a.e. on Ω_0. In particular, $|(\hat{f}/W)| = 2$ on the Shilov boundary for $H^\infty(W)$, but $|(\hat{f}/W)| \neq 2$ on $\tau(M(L^\infty(\Omega_0)))$.

Final result.

THEOREM 5. *Let $n \geqslant 2$ and $D \subset\subset \mathbf{C}^n$ be a strictly pseudoconvex domain with C^k boundary for $k > 2$. Then*

(1) *There is a closed nowhere dense subset β of $M(L^\infty(\mathrm{b}D))$ with measure $\hat{m}(\beta) = 0$ such that $\pi(\beta) \subset M(H^\infty(D))$ is a closed boundary for $H^\infty(D)$ where π is the map defined in the introduction.*

(2) *$\pi(M(L^\infty(\mathrm{b}D)))$ is strictly larger than the Shilov boundary for $H^\infty(D)$.*

(3) *The map $\pi\colon M(L^\infty(\mathrm{b}D)) \to M(H^\infty(D))$ is not one-to-one.*

PROOF. Cover $\mathrm{b}D$ with a collection of open neighborhoods of its points biholomorphically equivalent to ones of the origin in which $\mathrm{b}D$ can be represented as

$$x_1 = a_1(p) y_1^2 + \sum_{j=2}^{n} a_j(p) |z_j|^2 + \sum_{j=2}^{n} b_j(p) x_j y_1 + \sum_{j=2}^{n} c_j(p) y_j y_1 + O(3).$$

Since $\mathrm{b}D$ is compact it is possible to select a finite number of them so that $\mathrm{b}D = \bigcup_{l=1}^{\nu_0} V_l$. In each one of them consider the singularity sets $S^{(l)}$. They have measure zero in $\mathrm{b}D$. So if $\Sigma = \bigcup_{l=1}^{\nu_0} S^{(l)}$, it follows that $m(\Sigma) = 0$. Then the set $\mathrm{b}D - \Sigma$ is open and dense in $\mathrm{b}D$. Since $V_l - \Sigma = \bigcup_{r=1}^{\infty} \Omega_{l,r}$, it follows that

$$\mathrm{b}D - \Sigma = \bigcup_{l=1}^{\nu_0} \bigcup_{r=1}^{\infty} \Omega_{l,r} = \bigcup_{m=1}^{\infty} \Omega_m,$$

where the Ω_m's are a renumbering of the $\Omega_{l,r}$'s. For each one of them repeat the construction shown in Theorem 1 to obtain the sets $E_{k,m}$ and consider $E_k = \bigcup_{k=1}^{\infty} E_{k,m}$. Then E_k is open and dense in $\mathrm{b}D - \Sigma$ (i.e. in $\mathrm{b}D$) and $M(E_k) \leqslant dk^{-2}$, where d is a positive constant. Consider

$$U_k = \left\{ \psi \in M(L^\infty(\mathrm{b}D)) | \psi(\chi_{E_k}) - 1 \right\}.$$

$\pi(U_k)$ is closed in $M(H^\infty(D))$ and it is a closed boundary for $H^\infty(D)$. This follows from the properties of the sets $E_{k,m}$. Since $E_{k+1,m} \subset E_{k,m}$, it follows that $E_{k+1} \subset E_k$ and hence $U_{k+1} \subset U_k$ for all k. By compactness, then $\beta = \bigcap_{k=1}^{\infty} U_k$ is nonempty with $\hat{m}(\beta) = 0$; that is, β is nowhere dense in $M(L^\infty(\mathrm{b}D))$. Consider $\pi(\beta)$. It is a boundary for $H^\infty(D)$ because intersection of boundaries, indeed $\pi(\beta) = \bigcap_{k=1}^{\infty} \pi(U_k)$. This proves part (1) of the theorem.

The proof of the second statement is similar to the one for Theorem 4.

To prove the last part, let us recall that if $\lambda \in bD$, the fibers over λ are defined by

$$M_\lambda(L^\infty(bD)) = \left\{ \psi \in M(L^\infty(bD)) | \psi(z_j) = \lambda_j \text{ for } j = 1, \dots, n \right\}$$

and

$$M_\lambda(H^\infty(D)) = \left\{ \psi \in M(H^\infty(D)) | \psi(z_j) = \lambda_j \text{ for } j = 1, \dots, n \right\}$$

where z_j is the jth coordinate map. π and τ preserve fibers. By Theorem 3, the map τ: $M(L^\infty(\Omega_m)) \to M(H^\infty(W_m))$ is not one-to-one. Therefore, there exists a $\lambda \in \Omega_m$ such that τ: $M_\lambda(L^\infty(\Omega_m)) \to M_\lambda(H^\infty(W_m))$ is not one-to-one. $W_m = D \cup U$ for some open neighborhood U of λ in C^n. Then there are natural homeomorphisms between the fibers $M_\lambda(H^\infty(D))$ and $M_\lambda(H^\infty(W_m))$; $M_\lambda(L^\infty(\Omega_m))$ and $M_\lambda(L^\infty(bD))$ [**9**]. Let h_1 and h_2 be these homeomorphisms. Since the diagram

$$\begin{array}{ccc} M_\lambda(L^\infty(bD)) & \xrightarrow{\pi} & M_\lambda(H^\infty(D)) \\ \uparrow h_1 & & \uparrow h_2 \\ M_\lambda(L^\infty(\Omega_m)) & \xrightarrow{\tau} & M_\lambda(H^\infty(W_m)) \end{array}$$

commutes, it follows that π is not one-to-one. This proves the theorem.

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