

## ON THE SPECTRA OF $C_{11}$ -CONTRACTIONS

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ABSTRACT. We give a complete characterization of the closed subsets of the complex plane that can serve as spectra of completely nonunitary contractions of class  $C_{11}$ .

Let  $\mathcal{H}$  be a complex Hilbert space, and denote by  $\mathcal{L}(\mathcal{H})$  the algebra of all (continuous, linear) operators acting on  $\mathcal{H}$ . An operator  $T \in \mathcal{L}(\mathcal{H})$  is a contraction if  $\|T\| \leq 1$ . A contraction  $T \in \mathcal{L}(\mathcal{H})$  is said to be a  $C_{11}$ -contraction if  $\lim_{n \rightarrow \infty} \|T^n x\| \neq 0$  and  $\lim_{n \rightarrow \infty} \|T^{*n} x\| \neq 0$  for every nonzero vector  $x \in \mathcal{H}$ . Further, we recall that a contraction  $T$  is said to be completely nonunitary if it has no proper reducing subspace on which it acts as a unitary operator.

Our main result contains, in particular, a description of all spectra  $\sigma(T)$ , with  $T$  a completely nonunitary  $C_{11}$ -contraction. Before stating our results we need a few additional preliminaries.

We recall that the operators  $T \in \mathcal{L}(\mathcal{H})$  and  $T' \in \mathcal{L}(\mathcal{H}')$  are said to be quasisimilar if there exist operators  $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$  and  $Y \in \mathcal{L}(\mathcal{H}', \mathcal{H})$  with dense ranges and trivial kernels satisfying the relations  $T'X = XT$  and  $TY = YT'$ . It is known (cf. [10, Proposition II.3.5]) that each completely nonunitary  $C_{11}$ -contraction  $T$  is quasisimilar to an absolutely continuous unitary operator  $U$ ;  $U$  is determined by  $T$  up to unitary equivalence and there is in fact a canonical choice for  $U$ . Namely,  $T$  is quasisimilar to the residual part  $R_T$  of the minimal unitary dilation of  $T$ . We also have  $\sigma(R_T) \subset \sigma(T)$  (cf. [10, Proposition II.6.2]).

We will denote  $\mathbf{D} = \{\lambda \in \mathbf{C}: |\lambda| < 1\}$ ,  $\mathbf{T} = \{\lambda \in \mathbf{C}: |\lambda| = 1\}$ ,  $\mathbf{D}^- = \mathbf{D} \cup \mathbf{T}$ , and  $m$  will stand for the normalized Lebesgue measure on  $\mathbf{T}$  (i.e.,  $m(\mathbf{T}) = 1$ ). A closed subset  $\Sigma \subset \mathbf{T}$  will be called regular if  $\Sigma$  coincides with the closed support of the measure  $\chi_\Sigma dm$ , where, of course,  $\int_\omega \chi_\Sigma dm = m(\omega \cap \Sigma)$  for every Borel subset  $\omega$  of  $\mathbf{T}$ . If  $\Sigma$  is an arbitrary Borel subset of  $\mathbf{T}$ , we denote by  $\Sigma^-$  the closed support of the measure  $\chi_\Sigma dm$ . Note that  $\Sigma^-$  is regular and, in general, the measures  $\chi_\Sigma dm$  and  $\chi_{\Sigma^-} dm$  are different. The reason for introducing regular closed sets is that the spectrum of an absolutely continuous unitary operator is regular.

To be more specific, denote by  $L^2$  the space  $L^2(\mathbf{T}, dm)$  and, if  $\Sigma \subset \mathbf{T}$  is a Borel set with  $m(\Sigma) > 0$ , denote by  $L^2(\Sigma)$  the space of those (classes of) functions  $f \in L^2$  that vanish almost everywhere on  $\mathbf{T} \setminus \Sigma$ . The unitary operator  $M_\Sigma \in \mathcal{L}(L^2(\Sigma))$  is defined by

$$(M_\Sigma f)(\zeta) = \zeta f(\zeta), \quad \zeta \in \mathbf{T}, f \in L^2(\Sigma).$$

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It is well known (cf. [2]) that every cyclic, absolutely continuous, unitary operator is unitarily equivalent to an operator of the form  $M_\Sigma$ . Furthermore, we have  $\sigma(M_\Sigma) = \Sigma^\circ$  but, as noted above,  $M_\Sigma \neq M_{\Sigma^\circ}$  in general. (Let, e.g.,  $\Sigma$  be the complement of a Cantor-type set, and  $0 < m(\Sigma) < 1$ . In this case it may happen that  $\Sigma^\circ = \mathbf{T}$ .)

We can now state our main results. A subset  $\sigma'$  of a closed set  $\sigma \subset \mathbf{C}$  is called a clopen set if both  $\sigma'$  and  $\sigma \setminus \sigma'$  are closed.

(1) **THEOREM.** *Assume that  $T$  is a completely nonunitary  $C_{11}$ -contraction. Then  $m(\sigma' \cap \mathbf{T}) > 0$  for every nonempty clopen part  $\sigma'$  of  $\sigma(T)$ . More precisely, we have  $m(\sigma' \cap \sigma(R_T)) > 0$  for every nonempty clopen part  $\sigma'$  of  $\sigma(T)$ .*

We will use the sign “ $\simeq$ ” to indicate the unitary equivalence of operators.

(2) **THEOREM.** *Assume that  $\sigma \subset \mathbf{D}^-$  is a closed set,  $\Sigma \subset \sigma \cap \mathbf{T}$  is a regular closed set,  $\Sigma_0 \subset \Sigma$  is a Borel set,  $\Sigma_0^\circ = \Sigma$ , and  $m(\sigma' \cap \Sigma) > 0$  for every nonempty clopen part  $\sigma'$  of  $\sigma$ . Then there exists a completely nonunitary  $C_{11}$ -contraction  $T$  with the following properties: (i)  $T$  has a cyclic vector; (ii)  $\sigma(T) = \sigma$ ; (iii)  $\sigma(R_T) = \Sigma$ ; and (iv)  $R_T \simeq M_{\Sigma_0}$ .*

Observe that Theorem 2 is more than a converse to Theorem 1. We see, in particular, that the spectrum of a completely nonunitary  $C_{11}$ -contraction is also the spectrum of a certain cyclic  $C_{11}$ -contraction.

The question of determining the spectra of  $C_{11}$ -contractions was first posed by Sz.-Nagy and Foiaş in [9]; they also gave an example of a  $C_{11}$ -contraction  $T$  of infinite multiplicity with  $\sigma(T) = \mathbf{D}^-$  (cf. [10, Chapter VI.4.2]). Later Eckstein gave in [3] an example of a  $C_{11}$ -contraction  $T$  of infinite multiplicity with  $\sigma(T) = \{\lambda \in \mathbf{C} : \frac{1}{2} \leq |\lambda| \leq 1\}$ . The first examples of cyclic  $C_{11}$ -contractions  $T$  with  $\sigma(T) = \mathbf{D}^-$  were given in [1] and, while the present work was in progress, K. Takahashi found  $C_{11}$ -contractions  $T$  such that  $\sigma(T) = h(\mathbf{D})^-$ , where  $h \in H^\infty$ ,  $\|h\|_\infty \leq 1$ , and  $|h(e^{it})| = 1$  on a set of positive Lebesgue measure.

We begin now with the proof of our results.

**PROOF OF THEOREM 1.** Assume that  $T \in \mathcal{L}(\mathcal{H})$ , let  $\sigma'$  be a clopen part of  $\sigma(T)$ , and set  $\sigma'' = \sigma(T) \setminus \sigma'$ . By the Riesz-Dunford functional calculus (cf. [8, §148]) there exist invariant subspaces  $\mathcal{H}'$  and  $\mathcal{H}''$  for  $T$  such that  $\mathcal{H}' \cap \mathcal{H}'' = \{0\}$ ,  $\mathcal{H}' + \mathcal{H}'' = \mathcal{H}$ ,  $\sigma(T|_{\mathcal{H}'}) = \sigma'$ , and  $\sigma(T|_{\mathcal{H}''}) = \sigma''$ . The operator  $T$  is then similar to  $(T|_{\mathcal{H}'}) \oplus (T|_{\mathcal{H}''})$  and hence  $T' = T|_{\mathcal{H}'}$  is also a  $C_{11}$ -contraction. As noted above,  $T'$  is quasisimilar to the absolutely continuous unitary operator  $R_{T'}$ , and  $\sigma(R_{T'}) \subset \sigma'$ . Furthermore,  $\sigma' \neq \emptyset$  implies that  $\sigma(R_{T'}) \neq \emptyset$  and we conclude that

$$m(\sigma' \cap \mathbf{T}) \geq m(\sigma(R_{T'})) > 0.$$

The last statement of the theorem follows from the fact that  $\sigma(R_{T'}) \subset \sigma(R_T)$ . Indeed,  $R_T$  is quasisimilar to  $T$  and hence to  $R_{T|_{\mathcal{H}'}} \oplus R_{T|_{\mathcal{H}''}}$ . Thus  $R_T$  and  $R_{T'} \oplus R_{T|_{\mathcal{H}''}}$  are unitarily equivalent (cf. [10, Proposition II.3.4]). The theorem follows.

For the proof of Theorem 2 we need two lemmas. The first ingredient in our construction of  $C_{11}$ -contractions is provided by functional models.

(3) LEMMA. Let  $\alpha \subset \mathbf{T}$  be a Borel set with  $m(\alpha) > 0$ , and  $K$  a positive number. There exists a completely nonunitary  $C_{11}$ -contraction such that

- (i)  $T$  has a cyclic vector;
- (ii)  $\sigma(T) = \alpha^-$ ,  $R_T \simeq M_\alpha$ ; and
- (iii)  $\|T^{-1}\| \geq K$ .

PROOF. Choose a positive constant  $\gamma < 1$  and find an outer function  $\theta \in H^\infty$  such that

$$\begin{aligned} |\theta(\zeta)| &= 1 && \text{for almost every } \zeta \in \mathbf{T} \setminus \alpha, \\ &= \gamma && \text{for almost every } \zeta \in \alpha. \end{aligned}$$

Of course, such a function is essentially uniquely determined, and it is given by

$$\theta(\lambda) = \exp \left\{ \frac{1}{2\pi} \int_{\alpha} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log \gamma \, dt \right\}, \quad \lambda \in \mathbf{D}.$$

Define now  $T$  as the completely nonunitary contraction whose characteristic function coincides with  $\theta$ . Then  $T$  is a  $C_{11}$ -contraction because  $\theta$  is outer (cf. [10, Proposition VI.3.5]),  $T$  is similar to  $R_T$  by [6, Lemma 4], and  $R_T \simeq M_\alpha$  (cf. the proof of [10, Theorem VI.6.1]). Thus  $\sigma(T) = \sigma(M_\alpha) = \alpha^-$ , and (ii) holds. Now, (i) follows because  $T$  is similar to  $R_T$ , and  $R_T \simeq M_\alpha$  has a cyclic vector. Finally, it follows from the definition of characteristic functions [10, §VI.1] that  $\|T^{-1}\| \geq |\theta(0)|^{-1} = \exp(-m(\alpha)\log \gamma)$ . Thus condition (iii) is satisfied if  $\gamma$  is chosen to be sufficiently small. (We remark that, in fact,  $\|T^{-1}\| = |\theta(0)|^{-1}$ , but this equality is not needed here.)

The following result contains a basic idea that helps us produce  $C_{11}$ -contractions with contorted spectra. Since the proof of Lemma 3 actually produces similarities, one could replace “quasimilarity” by “similarity” in the following proof.

(4) LEMMA. Assume that we are given a simply connected domain  $\Omega \subset \mathbf{D}$  bounded by a simple, closed, rectifiable Jordan curve  $\Gamma$ , an arc  $I \subset \Gamma \cap \mathbf{T}$ , a Borel set  $\alpha \subset I$  with  $m(\alpha) > 0$ , a point  $\mu_0 \in \Omega$ , and a positive number  $K$ . Then there exists a completely nonunitary  $C_{11}$ -contraction  $T$  with the following properties:

- (i)  $T$  has a cyclic vector;
- (ii)  $\sigma(T) = \alpha^-$ ,  $R_T \simeq M_\alpha$ ;
- (iii)  $\|(\mu_0 I - T)^{-1}\| \geq K$ ; and
- (iv)  $\|(\mu I - T)^{-1}\| \leq 1/\text{dist}(\mu, \Omega^-)$  for  $\mu \in \mathbf{C} \setminus \Omega^-$ .

PROOF. First we note that there exists a homeomorphism  $u: \mathbf{D}^- \rightarrow \Omega^-$  such that  $u|_{\mathbf{D}}$  is holomorphic and  $u(0) = \mu_0$ . Indeed, the restriction  $u|_{\mathbf{D}}$  can be constructed by the Riemann mapping theorem, while the extendability of  $u$  to  $\mathbf{D}^-$  follows from a theorem of Carathéodory (cf. [5, Theorem II.4]) because  $\Gamma$  is a simple Jordan curve. Moreover, a subset  $\sigma \subset \mathbf{T}$  has Lebesgue measure zero if and only if  $u(\sigma)$  has arclength measure zero (cf. [5, X.1, Theorem 2]) and this clearly implies that the set

$\beta = u^{-1}(\alpha) \subset \mathbf{T}$  has positive Lebesgue measure, and

$$(5) \quad \beta^- = u^{-1}(\alpha^-) \subset \mathbf{T}.$$

Observe that there exists a function  $v \in H^\infty$  such that

$$(6) \quad \mu_0 - u(\lambda) = u(0) - u(\lambda) = \lambda v(\lambda), \quad \lambda \in \mathbf{D}.$$

Apply now Lemma 3 to produce a completely nonunitary  $C_{11}$ -operator  $T_1$  such that  $T_1$  has a cyclic vector,  $\sigma(T_1) = \beta^-$ ,  $R_{T_1} \simeq M_\beta$ , and

$$(7) \quad \|T_1^{-1}\| \geq K\|v\|_\infty.$$

Finally we define the required operator  $T$  by

$$T = u(T_1),$$

where the symbol  $u(T_1)$  is defined via the Sz.-Nagy-Foiaş functional calculus;  $T$  is completely nonunitary by [10, Theorem III.2.1]. The equality

$$\sigma(T) = u(\sigma(T_1)) = u(\beta^-) = \alpha^-$$

follows from (5) via [4, Corollary 3.2]. Next, since  $T_1$  is quasisimilar to  $M_\beta$ , it follows that  $T = u(T_1)$  is quasisimilar to  $u(M_\beta)$ . An application of [10, Theorem III.2.3] shows that  $u(M_\beta)$  acts as follows:

$$(u(M_\beta)f)(\zeta) = u(\zeta)f(\zeta), \quad \zeta \in \mathbf{T}, f \in L^2(\beta),$$

and it is easy to see that this operator is unitarily equivalent to  $M_\alpha$ . An explicit unitary equivalence  $U: L^2(\alpha) \rightarrow L^2(\beta)$  is provided by the formula

$$\begin{aligned} (Ug)(\zeta) &= |u'(\zeta)|^{1/2} g(u(\zeta)), & \zeta \in \beta, \\ &= 0, & \zeta \in \mathbf{T} \setminus \beta, \end{aligned}$$

and this makes sense for  $g \in L^2(\alpha)$  because of the behavior of  $u$  with respect to sets of measure zero (cf. [5, §X.1, Theorem 1]).

We see now that  $T$  is quasisimilar to the unitary operator  $u(M_\beta) \simeq M_\alpha$ , and this finishes the proof of (ii). That  $T$  is a  $C_{11}$ -contraction and has a cyclic vector follows from the facts that  $M_\alpha$  is unitary and has a cyclic vector.

Observe now that (6) implies  $\mu_0 I - T = u(0)I - u(T_1) = T_1 v(T_1)$ , from which we deduce

$$v(T_1)(\mu_0 I - T)^{-1} = T_1^{-1},$$

and hence, by (7),

$$K\|v\|_\infty \leq \|T_1^{-1}\| \leq \|v(T_1)\| \|(\mu_0 I - T)^{-1}\| \leq \|v\|_\infty \|(\mu_0 I - T)^{-1}\|.$$

The inequality (iii) follows at once. As for (iv), Theorem III.2.1 of [10] implies that

$$(\mu I - T)^{-1} = v_\mu(T_1), \quad \mu \notin \Omega^-,$$

where  $v_\mu(\lambda) = 1/(\mu - u(\lambda))$ ,  $\lambda \in \mathbf{D}$ , whence

$$\|(\mu I - T)^{-1}\| \leq \|v_\mu\|_\infty = 1/\text{dist}(\mu, \Omega^-).$$

The lemma is proved.

PROOF OF THEOREM 2. Choose a sequence  $\{\lambda_n: n \geq 1\}$  dense in  $\sigma$ , and let  $\{\mu_n: n \geq 1\}$  be a sequence in which every  $\lambda_i$  is repeated infinitely many times,  $i \geq 1$ . The idea is to construct a sequence of pairwise disjoint Borel sets  $\{\alpha_n: n \geq 1\}$ ,  $\alpha_n \subset \Sigma_0$ , and a sequence of completely nonunitary  $C_{11}$ -contractions  $\{T_n: n \geq 1\}$  with the following properties:

$$(8) \quad R_{T_n} \simeq M_{\alpha_n} \quad \text{for } n \geq 1;$$

$$(9) \quad \sigma(T_n) = \alpha_n^- \quad \text{for } n \geq 1;$$

$$(10) \quad \|(\mu_n I - T_n)^{-1}\| \geq n \quad \text{if } (\mu_n I - T_n)^{-1} \text{ exists, } n \geq 1;$$

and

$$(11) \quad \|(\mu I - T_n)^{-1}\| \leq 1/[\text{dist}(\mu, \sigma) - 1/n] \quad \text{if } \text{dist}(\mu, \sigma) > 1/n, n \geq 1.$$

Once we have constructed these sequences we set  $\alpha_0 = \Sigma_0 \setminus \bigcup_{n=1}^{\infty} \alpha_n$ , and define a completely nonunitary  $C_{11}$ -contraction  $T_0$  such that  $\sigma(T_0) \subset \alpha_0^-$ , and  $R_{T_0} \simeq M_{\alpha_0}$ ; note that  $T_0$  acts on a trivial space if  $m(\alpha_0) = 0$ . Finally, define  $T = \bigoplus_{n=0}^{\infty} T_n$ , and note that  $T$  is a completely nonunitary  $C_{11}$ -contraction, and

$$R_T \simeq \bigoplus_{n=0}^{\infty} R_{T_n} \simeq \bigoplus_{n=0}^{\infty} M_{\alpha_n} \simeq M_{\Sigma_0}.$$

Thus, as before,  $T$  has a cyclic vector and  $\sigma(R_T) = \Sigma$ . In order to check (ii) we observe that for fixed  $i \geq 1$  we have  $\|(\lambda_i I - T)^{-1}\| \geq n$  for infinitely many values of  $n$ , and hence  $\lambda_i \in \sigma(T)$ . By density we conclude that  $\sigma \subset \sigma(T)$ . Furthermore, if  $\mu \notin \sigma$  then  $\mu \notin \sigma(T_n) = \alpha_n^-$ , and (11) shows that the sequence  $\{\|(\mu I - T_n)^{-1}\|: n \geq 1\}$  is bounded, consequently  $\mu \notin \sigma(T)$ . We conclude that  $\sigma(T) = \sigma$  and hence  $T$  satisfies all the requirements of the theorem.

We turn now to the construction of  $\alpha_n$  and  $T_n$ . For  $n \geq 1$  we set

$$G_n = \{\lambda \in \mathbf{C}: \text{dist}(\lambda, \sigma) < 1/n\}$$

and denote by  $G'_n$  the connected component of  $G_n$  that contains  $\mu_n$ . Also choose for each  $n$  a point  $\mu'_n \in G'_n \cap \mathbf{D}$  such that

$$(12) \quad |\mu_n - \mu'_n| < 1/2n.$$

Of course, we may choose  $\mu_n = \mu'_n$  if  $|\mu_n| < 1$ . The set  $G'_n \cap \sigma$  is clearly nonempty (because  $\mu_n \in G'_n$ ) and clopen in  $\sigma$ . We conclude that  $m(G'_n \cap \Sigma) > 0$  and, since  $\Sigma_0^- = \Sigma$ ,

$$m(G'_n \cap \Sigma_0) > 0.$$

Thus we can find inductively Borel subsets  $\beta_n \subset G'_n \cap \Sigma_0$  such that

$$0 < m(\beta_n) \leq \frac{1}{3}m(\beta_{n-1}), \quad n \geq 2.$$

It is clear that the set  $\alpha_n = \beta_n \setminus \bigcup_{k=1}^{\infty} \beta_{n+k}$  has positive measure. Indeed,

$$m(\alpha_n) \geq m(\beta_n) - \sum_{k=1}^{\infty} m(\beta_{n+k}) \geq m(\beta_n) - m(\beta_n) \sum_{k=1}^{\infty} 3^{-k} > 0, \quad n \geq 1.$$

We can further assume that each  $\alpha_n$  is contained in a single arc of  $G'_n \cap \mathbf{T}$ . The sets  $\{\alpha_n; n \geq 1\}$  are pairwise disjoint and  $m(\alpha_n) > 0$ ,  $n \geq 1$ . We concentrate now on the construction of  $T_n$  for  $n \geq 1$ .

Assume that  $\alpha_n$  is contained in the arc  $I_n = \{e^{it}; t_1 < t < t_2\} \subset G'_n \cap \mathbf{T}$  with endpoints  $\zeta_1 = e^{it_1}$  and  $\zeta_2 = e^{it_2}$ , where  $t_1 < t_2 < t_1 + 2\pi$ . A moment's thought shows that the set  $G'_n \cap \mathbf{D}$  is connected. The easiest way to see this is to note that, if a point  $\lambda$  with  $|\lambda| > 1$  belongs to  $G_n$ , then the point  $\lambda/|\lambda|$  belongs to the same component of  $G_n$ . Thus we can find a rectifiable, simple, Jordan curve

$$\Gamma_n \subset (G'_n \cap \mathbf{D}) \cup \{\zeta_1, \zeta_2\}$$

joining  $\zeta_1$  and  $\zeta_2$ , and such that the simply connected region  $\Omega_n$  with boundary  $I_n \cup \Gamma_n$  is entirely contained in  $G'_n \cap \mathbf{D}$ , and  $\mu'_n \in \Omega_n$ . We can now apply Lemma 4 with the domain  $\Omega_n$ , the point  $\mu'_n \in \Omega_n$ , the Borel set  $\alpha_n$ , and  $K = 3n$ . We obtain an operator  $T_n$  satisfying (8), (9),

$$(13) \quad \|(\mu'_n I - T_n)^{-1}\| \geq 3n,$$

and

$$(14) \quad \|(\mu I - T_n)^{-1}\| \leq 1/\text{dist}(\mu, \Omega_n), \quad \mu \in \mathbf{C} \setminus \Omega_n^-.$$

To conclude the proof we must show that (10) and (11) are also verified. Observe that for  $\mu \in \mathbf{C} \setminus \sigma$  we have

$$\begin{aligned} \text{dist}(\mu, \sigma) &\leq \text{dist}(\mu, G_n) + 1/n \\ &\leq \text{dist}(\mu, G'_n) + 1/n \leq \text{dist}(\mu, \Omega_n) + 1/n \end{aligned}$$

because  $\Omega_n \subset G'_n \subset G_n$ . Thus  $\text{dist}(\mu, \sigma) > 1/n$  implies that  $\mu \in \mathbf{C} \setminus \Omega_n^-$  and the inequality (11) holds. To prove (10) we note that (13) implies the existence of a unit vector  $x_n$ , in the space on which  $T_n$  acts, such that  $\|(\mu'_n I - T)x_n\| \leq 1/2n$ . Now, by (12),

$$\|(\mu_n I - T)x_n\| \leq \|(\mu'_n I - T)x_n\| + |\mu_n - \mu'_n| \leq 1/n$$

and this in turn implies (10). The proof is complete.

We conclude with a few remarks about the spectra of  $C_{11}$ -contractions that are not completely nonunitary. The results above would not change if the words "completely nonunitary contractions" are replaced by "contractions whose unitary parts are absolutely continuous." In fact, every contraction with an absolutely continuous unitary part is similar to some completely nonunitary contraction. Now, adding (orthogonally) a singular unitary operator to a given contraction allows us to increase the spectrum of that contraction by an arbitrary closed subset of  $\mathbf{T}$ . The resulting characterization of the spectra of arbitrary  $C_{11}$ -contractions is rather cumbersome, and the reader will have no difficulty in formulating it.

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