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WEAK SPECTRAL THEORY

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ABSTRACT. We initiate the weak spectrum of a linear operator on L^p spaces, $1 \leqslant p < \infty$. The weak spectrum of a pseudo-differential operator with symbol in $S^m_{\alpha\sigma}$, where $-\infty < m < \infty$ and $0 \leqslant \rho \leqslant 1$, is investigated.

1. Introduction. For $m \in (-\infty, \infty)$ and $\rho \in [0,1]$, we define $S^m_{\rho,0}$ to be the set of all functions σ in $C^\infty(\mathbf{R}^n)$ such that, for each multi-index α, $(D^\alpha\sigma)(\xi) = O(|\xi|^{m-\rho|\alpha|})$ as $|\xi| \to \infty$. Let $\sigma \in S^m_{\rho,0}$. Then we define the pseudo-differential operator T_σ, initially on $\mathscr{S}$(the Schwartz space), by

$$(T_\sigma\varphi)(x) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(\xi)\hat{\varphi}(\xi)\,d\xi$$

for all $\varphi \in \mathscr{S}$. Here, ^ denotes the Fourier transformation. Obviously T_σ maps $\mathscr{S}$ into $\mathscr{S}$. It can be shown (see Proposition 2.1 in Wong [5]) that, for $1 \leqslant p \leqslant \infty$, T_σ: $\mathscr{S} \to \mathscr{S}$ is closable in $L^p(\mathbf{R}^n)$. We denote the closure by $T_{\sigma p}$. Detailed information about the spectrum $\Sigma(T_{\sigma p})$ of $T_{\sigma p}$ can be found in Wong [**3**, **4**, **5**]. The corresponding results for partial differential operators have been gathered in Schechter [**2**].

2. The weak spectrum. Let A be a closed linear operator defined on $L^p(\mathbf{R}^n)$. Then a complex number λ is said to be in the weak resolvent set $\rho_w(A)$ of A if the range $R(A-\lambda)$ of $A-\lambda$ is dense in $L^p(\mathbf{R}^n)$ and there is a constant $C > 0$ such that

$$m\{x \in \mathbf{R}^n\colon |\varphi(x)| > \alpha\} \leqslant \{C\|(A-\lambda)\varphi\|/\alpha\}^p$$

for all $\alpha > 0$ and φ in the domain $\mathscr{D}(A)$ of A. Here, $m\{\cdots\}$ denotes the Lebesgue measure of $\{\cdots\}$ and $\|\ \|$ the L^p norm. As usual, the weak spectrum $\Sigma_w(A)$ of A is defined to be $\mathbf{C} - \rho_w(A)$. Obviously, $\Sigma_w(A) \subseteq \Sigma(A)$. That $\Sigma_w(A)$ can be a proper subset of $\Sigma(A)$ will be shown in §5.

3. On $\Sigma_w(T_{\sigma p})$, $1 \leqslant p < \infty$. We first show that $\Sigma_w(T_{\sigma p})$ is not empty.

THEOREM 3.1. *If $\sigma(\xi)$ is not bounded away from a complex number λ for all $\xi \in \mathbf{R}^n$, then $\lambda \in \Sigma_w(T_{\sigma p})$.*

PROOF. For simplicity, we suppose that $\lambda = 0$. Let $\{\xi_k\}$ be a sequence of elements of $\mathbf{R}^n$ such that $\sigma(\xi_k) \to 0$ as $k \to \infty$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers. Let $\theta \in C_0^\infty(\mathbf{R}^n)$ be such that $\theta(\xi) = 0$ for $|\xi| > 1$ and $(2\pi)^{-n/2}\int_{\mathbf{R}^n}\theta(\xi)\,d\xi = 1$. Let

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$\psi \in \mathcal{S}$ be such that $\hat{\psi} = \theta$. For $k = 1, 2, \ldots,$ define

$$\varphi_k(x) = \varepsilon_k^{n/p} \psi(\varepsilon_k x) e^{i \xi_k \circ x}. \tag{3.1}$$

If $0 \in \rho_w(T_{\sigma p})$, then there is a constant $C > 0$ such that

$$m\{x \in \mathbf{R}^n: |\varphi_k(x)| > \alpha\} \leqslant \{C \|T_\sigma \varphi_k\| / \alpha\}^p \tag{3.2}$$

for all $\alpha > 0$ and $k = 1, 2, \ldots.$ Choosing $\alpha = \frac{1}{2}\varepsilon_k^{n/p}$ and using (3.1), inequality (3.2) becomes

$$m\{x \in \mathbf{R}^n: |\psi(\varepsilon_k x)| > \tfrac{1}{2}\} \leqslant \varepsilon_k^{-n} O\left(\|T_\sigma \varphi_k\|^p\right). \tag{3.3}$$

Since $\psi(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \theta(\xi)\, d\xi = 1$, it follows that there exists a $\delta > 0$ such that $|\psi(x)| > \frac{1}{2}$ whenever $|x| \leqslant \delta$. Therefore

$$m\{x \in \mathbf{R}^n: |\psi(\varepsilon_k x)| > \tfrac{1}{2}\} \geqslant \pi(\delta/\varepsilon_k)^n$$

for $k = 1, 2, \ldots.$ Hence, by (3.3),

$$\pi \delta^n \leqslant O\left(\|T_\sigma \varphi_k\|^p\right) \tag{3.4}$$

for $k = 1, 2, \ldots.$ But as has been shown in the proof of Theorem 3.1 in Wong [**5**], we can choose the ε_k's going to zero so fast that $\|T_\sigma \varphi_k\| \to 0$ as $k \to \infty$. Thus (3.4) is impossible.

A useful consequence of Theorem 3.1 is

COROLLARY 3.2. *$\Sigma_w(T_{\sigma p})$ contains the set $\{\sigma(\xi): \xi \in \mathbf{R}^n\}$.*

4. Multipliers of weak type (p, p), $1 \leqslant p < \infty$. Let m be a bounded measurable function on $\mathbf{R}^n$. For any $\varphi \in \mathcal{S}$, we define $T_m \varphi$ by

$$(T_m \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix \cdot \xi} m(\xi) \hat{\varphi}(\xi)\, d\xi.$$

Suppose that there is a constant $C > 0$ such that

$$m\{x \in \mathbf{R}^n: |(T_m \varphi)(x)| > \alpha\} \leqslant \{C\|\varphi\|/\alpha\}^p$$

for all $\alpha > 0$ and $\varphi \in \mathcal{S}$. Then we call T_m a multiplier of weak type (p, p).

The connection between weak type multipliers and weak spectra is provided by

THEOREM 4.1. *A complex number λ is in the weak resolvent set $\rho_w(T_{\sigma p})$ of $T_{\sigma p}$ if and only if $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak type (p, p).*

PROOF. We first prove necessity. Again for simplicity, let $\lambda = 0$. By Theorem 3.1, $\sigma(\xi)$ is bounded away from 0 for all $\xi \in \mathbf{R}^n$. For any $f \in \mathcal{S}$, define u by $\hat{u}(\xi) = \hat{f}(\xi)/\sigma(\xi)$. Then $u \in \mathcal{S}$ and $T_\sigma u = f$. Since $0 \in \rho_w(T_{\sigma p})$, it follows that there is a constant $C > 0$ such that

$$m\{x \in \mathbf{R}^n: |u(x)| > \alpha\} \leqslant \{C\|f\|/\alpha\}^p$$

for all $\alpha > 0$ and $f \in \mathcal{S}$. Hence $1/\sigma(\xi)$ is a multiplier of weak type (p, p). Conversely, if $1/\sigma(\xi)$ is a multiplier of weak type (p, p), then there is a constant $C > 0$ such that

$$m\{x \in \mathbf{R}^n: |\varphi(x)| > \alpha\} \leqslant \{C\|T_\sigma \varphi\|/\alpha\}^p \tag{4.1}$$

for all $\alpha > 0$ and $\varphi \in \mathscr{S}$. Since $\sigma(\xi)$ is bounded away from 0 for all $\xi \in \mathbf{R}^n$, it follows that $\mathscr{S} \subseteq R(T_{\sigma p})$. This proves that $R(T_{\sigma p})$ is dense in $L^p(\mathbf{R}^n)$. Consequently, $0 \in \rho_w(T_{\sigma p})$ if we can show that (4.1) is valid for all $\varphi \in \mathscr{D}(T_{\sigma p})$.

LEMMA 4.2. *Inequality* (4.1) *is valid for all* $\varphi \in \mathscr{D}(T_{\sigma p})$.

PROOF. For any $\varphi \in \mathscr{D}(T_{\sigma p})$, let $\{\varphi_k\}$ be a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi_k \to \varphi$ and $T_\sigma \varphi_k \to T_{\sigma p}\varphi$ in $L^p(\mathbf{R}^n)$ as $k \to \infty$. Pick a subsequence of $\{\varphi_k\}$, again denoted by $\{\varphi_k\}$, such that $\varphi_k \to \varphi$ a.e. on $\mathbf{R}^n$. For any $\alpha > 0$, we set

$$E(\alpha) = \{x \in \mathbf{R}^n \colon |\varphi(x)| > \alpha\}$$

and

$$E_k(\alpha) = \{x \in \mathbf{R}^n \colon |\varphi_k(x)| > \alpha\}$$

for $k = 1, 2, \ldots$. Since $\varphi \in L^p(\mathbf{R}^n)$, it follows that $m(E(\alpha)) < \infty$. So for any $\varepsilon > 0$, by Egoroff's Theorem, we can find a measurable set $A_\varepsilon \subseteq \mathbf{R}^n$ such that $m(A_\varepsilon) < \varepsilon$ and $\varphi_k(x) \to \varphi(x)$ uniformly for all $x \in E(\alpha) - A_\varepsilon$. Hence, there exists a positive integer K such that $|\varphi_k(x)| > \alpha$ whenever $k \geqslant K$. For such k's, $E(\alpha) - A_\varepsilon \subseteq E_k(\alpha)$, and using (4.1) and letting $k \to \infty$, we get

$$m(E(\alpha)) - \varepsilon \leqslant \{C\|T_{\sigma p}\varphi\|/\alpha\}^p$$

for every $\alpha > 0$. Since ε is an arbitrary positive number, the proof is complete.

5. An example. We begin with an observation.

LEMMA 5.1. *For any p such that $1 \leqslant p < \infty$, a sufficient condition for $\Sigma_w(T_{\sigma p}) = \Sigma(T_{\sigma p})$ is that $\Sigma(T_{\sigma p}) = \{\sigma(\xi) \colon \xi \in \mathbf{R}^n\}$.*

Lemma 5.1 follows immediately from Corollary 3.2 and the fact that $\Sigma_w(T_{\sigma p}) \subseteq \Sigma(T_{\sigma p})$.

Let τ be the function defined by $\tau(\xi) = e^{i|\xi|^a}/(1 + |\xi|^c)$, where $0 < a < 1$ and $0 < c < na/2$. Then, defining σ by $\sigma(\xi) = 1/\tau(\xi)$, it is clear that $\sigma \in S^c_{1-a,0}$. As has been proved in Wong [3, 4], $\Sigma(T_{\sigma p}) = \{\sigma(\xi) \colon \xi \in \mathbf{R}^n\}$ if p is any number such that $1 < p < \infty$ and $|1/p - 1/2| \leqslant c/na$, and $\Sigma(T_{\sigma p}) = \mathbf{C}$ if $|1/p - 1/2| > c/na$. The following result tells us that we know exactly what $\Sigma_w(T_{\sigma p})$ is if $1 < p < \infty$.

THEOREM 5.2. $\Sigma_w(T_{\sigma p}) = \Sigma(T_{\sigma p})$, $1 < p < \infty$.

PROOF. In view of Lemma 5.1, we need only consider $|1/p - 1/2| > c/na$. We first suppose that $1/p > 1/2 + c/na$. If $\lambda \in \Sigma(T_{\sigma p})$ and $\lambda \notin \Sigma_w(T_{\sigma p})$, then by Theorem 4.1, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak types $(2, 2)$ and (p, p). Hence, by the Marcinkiewicz interpolation theorem, $1/(\sigma(\xi) - \lambda)$ is an L^q multiplier for any q such that $1/2 + c/na < 1/q < 1/p$. Thus, by Theorem 3.3 in Wong [3], $\lambda \in \rho(T_{\sigma q})$. But Theorem 3.1 in Wong [4] says that the spectrum of $T_{\sigma q}$ is either $\mathbf{C}$ or $\{\sigma(\xi) \colon \xi \in \mathbf{R}^n\}$. Hence, $\Sigma(T_{\sigma q}) = \{\sigma(\xi) \colon \xi \in \mathbf{R}^n\}$. This is a contradiction. The proof for the case when $1/p < 1/2 - c/na$ is similar.

THEOREM 5.3. $\Sigma_w(T_{\sigma 1}) = \{\sigma(\xi) \colon \xi \in \mathbf{R}^n\}$.

PROOF. By Theorems 3.1 and 4.1, we need only show that if $\lambda \in \mathbf{C}$ is such that $\sigma(\xi) \neq \lambda$ for all $\xi \in \mathbf{R}^n$, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak type $(1, 1)$. But an easy computation shows that $1/(\sigma(\xi) - \lambda) \in S^{c}_{1-a,0}$, and hence it follows from Theorem 2′ in Fefferman [**1**] that $1/(\sigma(\xi) - \lambda)$ is indeed a multiplier of weak type $(1, 1)$.

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